



Experiencing

EXPERIENCING
mathematics



Experiencing EXPERIENCING mathematics

This virtual exhibition is addressed to math teachers and their students - especially those in secondary education - but also to anyone interested in mathematics and Sciences in general.

This virtual exhibition shows **more than 200 mathematical situations**. They give students opportunities to experiment, try out, make hypotheses, probe them, try to validate them but also the possibility to try to demonstrate and debate about mathematical properties.

As for the touring international exhibition «**why mathematics?**», it was initiated and supported by UNESCO, designed and carried out by Centre Sciences and Adecum.

As for «**Experiencing Mathematics**», this exhibition offers «**off-the-table**» experiments, easy to carry out with little simple material: your head and hands, paper and pens, cardboard, wood or acrylic sheets, wire and nails... Thanks to its numeric approach, the exhibition also offers «**off-the-screen**» experiments where one can perform the experiments with a simple click

For each theme, you will find:

- an attractive interactive introduction,
- experiments to be carried out by the students,
- some explanations and historical references,
- some career perspectives related to these math,
- search keywords for the Internet,
- a few tips in a pdf file to print out.

For it is addressed especially to teachers of Southern countries, this exhibition, although virtual, is definitely not meant to increase the **digital divide**. This is why it gives the opportunity to those who have neither a computer nor an Internet connection the possibility to print or have the pages printed – in black and white or in colour – from the **pdf files** especially setup for this purpose..

* Centre•Sciences: centre of scientific and technical culture of region Centre – Orléans

* Adecum: association for the development of mathematical culture – Orléans

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1. Read the Nature



Spirals in nature



A world fractal



Conics in Space

1. Read the Nature

Spirals in Nature



Do it yourself

MATERIAL: 1 felt pen, 1 pineapple, 1 pine cone, 1 strawberry, 1 sunflower head ...

How many spirals are therein each direction ?



1, 1, 2, 3, 5, 8...

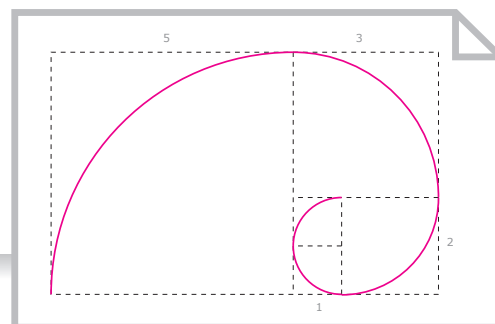
- Find the following elements of this sequence.
- Look at one of these objects. You may observe spirals winding in opposite directions.
- Now, count the number of spirals.
- Find other examples of fruits, flowers or leaves with the same

Do it yourself

MATERIAL: 1 sheand of grid paper, 1 ruler, 1 pair of compasses, 1 pencil, 1 pair of scissors

Draw a golden spiral

- Take a sheand of grid paper and draw squares with sides 1, 1, 2, 3, 5, 8...
- Then, draw an arc from one corner of each square as indicated here.
- Finally, cut the squares out and organize them into a spiral as described below.



To notice

**1, 1, 2, 3, 5, 8, 13,
21, 34, 55, 89, ...,
233, ...**

The seeds of some fruits, the pandals of some flowers, the leaves of some trees, divide up according to the same sequence of numbers: each number after the second is the sum of the preceding two. This sequence is often observed in nature. So, in a pine cone, a pineapple, a sunflower head, the spiral sequences are numbers of the **Fibonacci** sequence. This feature may also be observed in the lengths of the tangents of the chambered nautilus.

Going further

Fibonacci numbers are a sequence of integers where each term equals the sum of the preceding two. This sequence was discovered 8 centuries ago by an Italian called Leonardo of Pisa, also known as Fibonacci.

Take F_n as the number of rank n for this sequence.

This sequence has many interesting properties :

F_{n+1} / F_n tends towards the limit: $(1 + \sqrt{5})/2$ which is called the golden section.
 F_n and F_{n+1} are coprime (no common denominator) and the sum of their squares is found in the Fibonacci sequence :
 $(F_n)^2 + (F_{n+1})^2 = F_{2n+1}$
 ...

Maths-related jobs

Ever since **Fibonacci**, many people got interested in these properties which can be found in plants. Recently, **Stéphane Douady** and **Yves Couder**, two french researchers, have shown, thanks to dynamic systems in physics, that plants growth follows these properties. The study of the arrangement of elements in plants is called phyllotaxis or phyllotaxy. Botanists and biologists are very much interested in phyllotaxis.

WEBSITES:

<http://www.lps.ens.fr/~douady/>
<http://maven.smith.edu/~phyllo/>

KEYWORDS FOR WEBSITES:

Fibonacci - Sequences - Phyllotaxis - Golden section - Golden spiral

A world fractal



Do it yourself

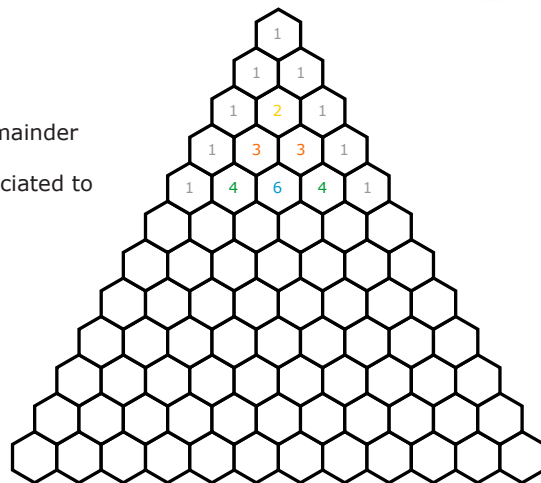
MATERIAL: 1 square or hexagonal grid, 4 or 5 coloured pencils

10

11

Pascal's triangle in colour

- Continue the grid following the setup rule.
- Associate 3 different colours to 0, 1 and 2.
- Replace the number in each box by the remainder of the Euclidean division by 3.
- Then, colour each box with the colour associated to the remainder of the division (0, 1 or 2).



A world fractal



Do it yourself

MATERIAL: 1 sheet of paper, 1 pencil, 1 ruler

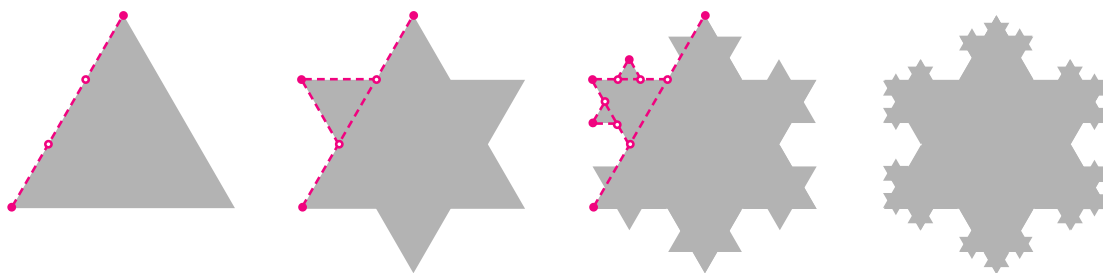
Draw more sequences of fractal figures

- Draw an equilateral triangle.
- Cut each side into 3 equal segments, then replace the middle segment by two equal ones.
- Repeat the process for each new segment.

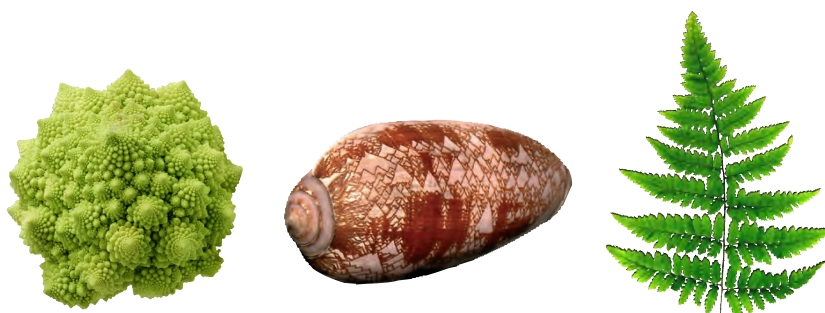
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Further activity :

at each stage, calculate the perimeter and the surface area, then their limits



What about in Nature ?



Maths-related jobs

Fractal objects appear or are used in numerous domains : meteorology, economy, pictures compression, medicine and even Art... fractal art !

WEBSITE:

<http://commons.wikimedia.org/wiki/Fractal>

KEYWORDS FOR WEBSITES:

Fractal - Fractal figures sequences - Fractal dimension - Mandelbrot

Conics in Space

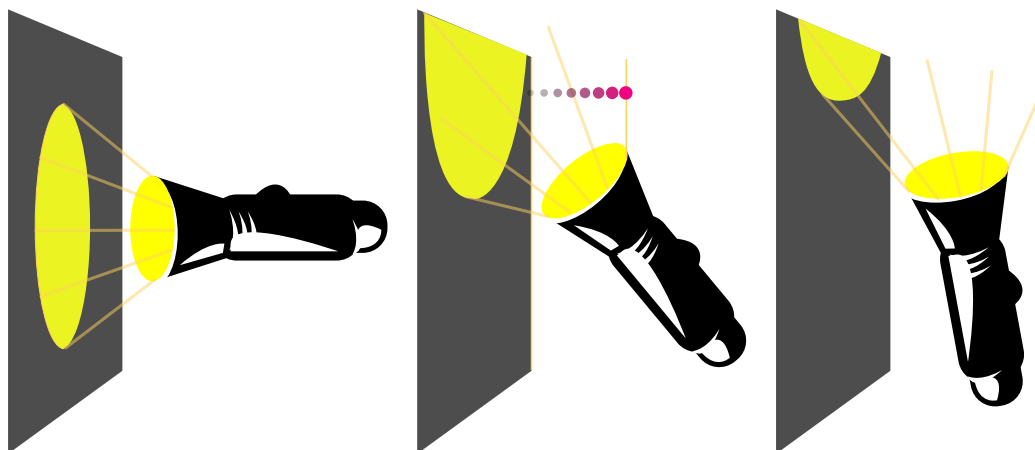


Do it yourself

MATERIAL: 1 lamp or 1 torch, 1 white wall or 1 white screen

Lights & Conics

- Use the lamp to light up the wall or the screen.
- A spot of light appears.
- What shape is it ? Can you change it ? How ?

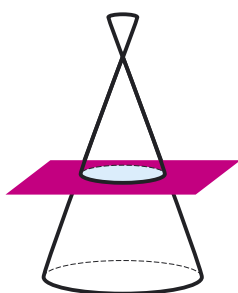


To notice

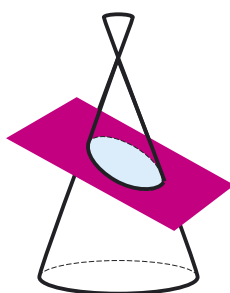
On the ceiling or the floor, you may observe a circle or an ellipse. On the walls, or if the lamp is tilted, you may observe part of a parabola or a hyperbola.

Conics are curves getting from intersection by a cone with a plan. Depending on the slope of the plane and its position relative to the axis of the cone, you may obtain different types of conics.

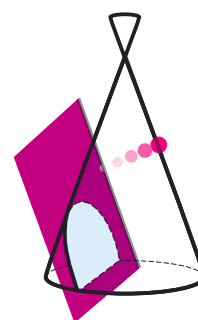
When the axis is perpendicular to the screen, you obtain a circle. As the angle changes, you obtain an ellipse, a parabola (when one side of the light cone is parallel to the screen) and finally, one or two branches of hyperbola. Can we get one point, one line or two ?



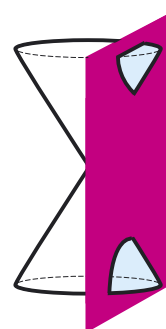
CIRCLE



ELLIPSE



PARABOLA



HYPERBOLA

Conics in Space



Do it yourself

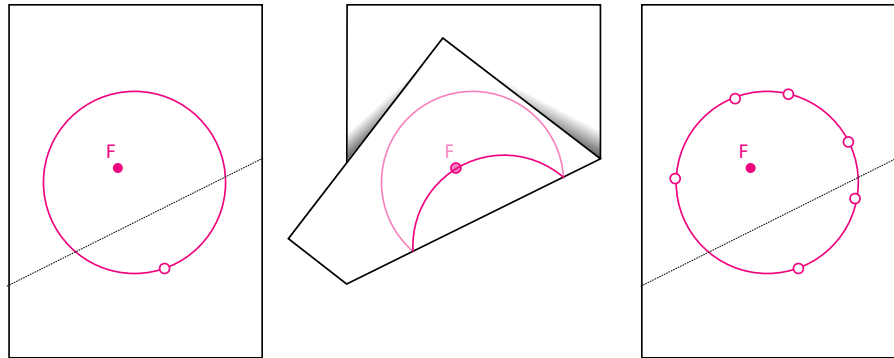
MATERIAL: 1 sheet of paper, 1 ruler, 1 pair of compasses, 1 pen

Getting conic sections by folding

- Draw a circle and mark a spot F, inside or outside the circle.
- Fold the paper (and mark the fold) so that F meets an edge of the circle.
- Repeat the process over twenty times.
- What do you see ?

Remember

- If F is inside the circle, the folds surround an ellipse.
 - If F is outside the circle, the folds create a hyperbola.
- What we get if we replace the circle by a line?



Going further

Conics may be observed in many natural phenomena.

Parabola: from a fountain to the trajectory of an object away from you, but also the car headlights and solar cookers.

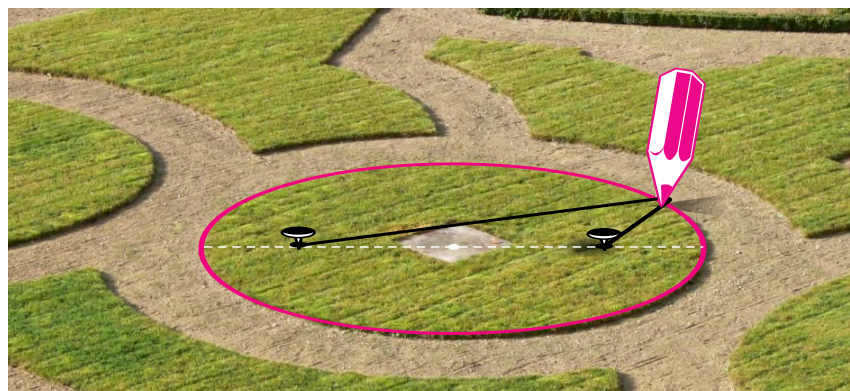
Ellipses are found in architecture and in perspective drawings of circle.

The laws of **Kepler** (1619) then the law of gravitation described by **Newton** (1687), show that orbits of celestial bodies, natural or not, are conics.

Maths-related jobs

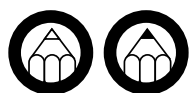
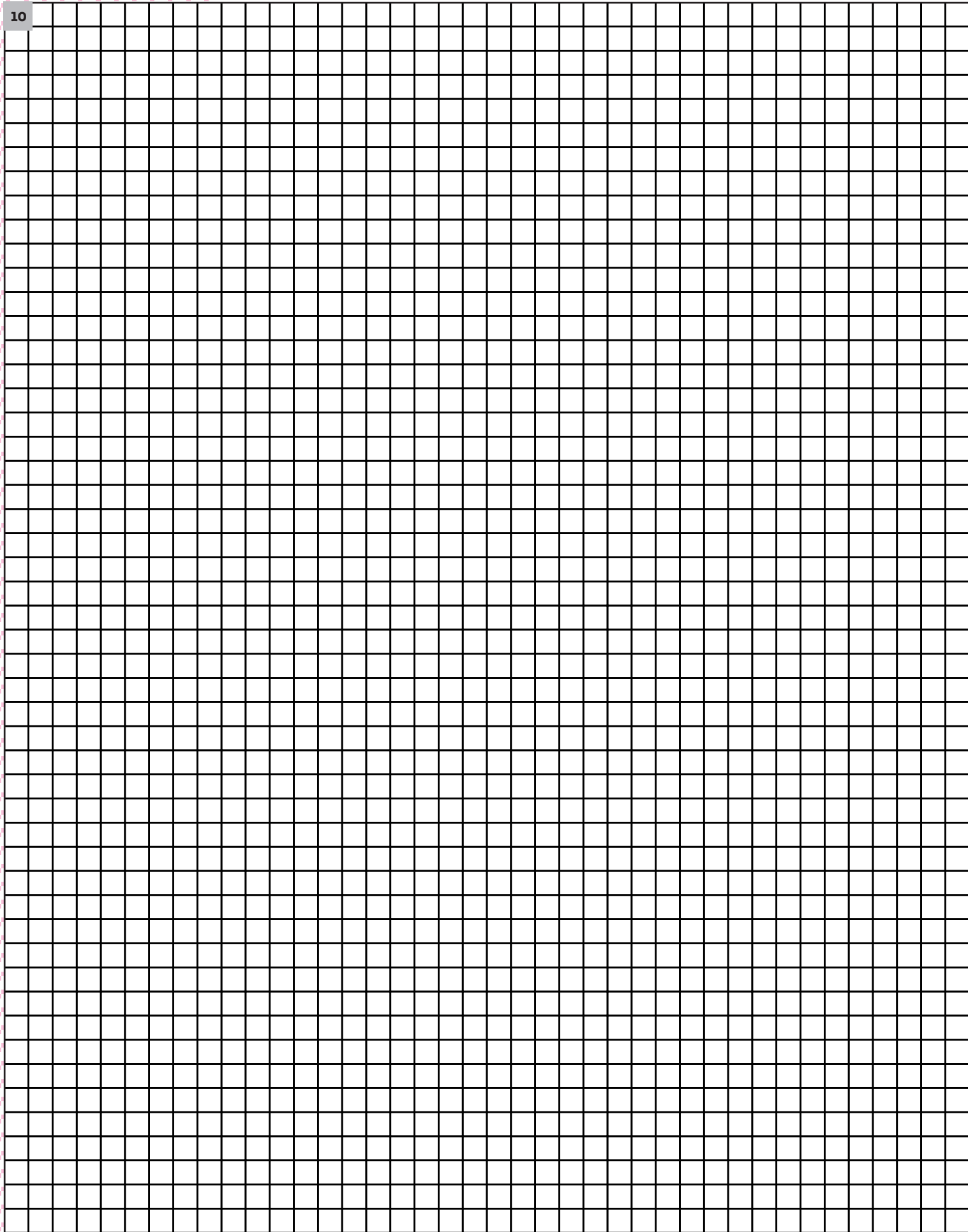
Who uses conics ?

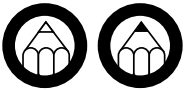
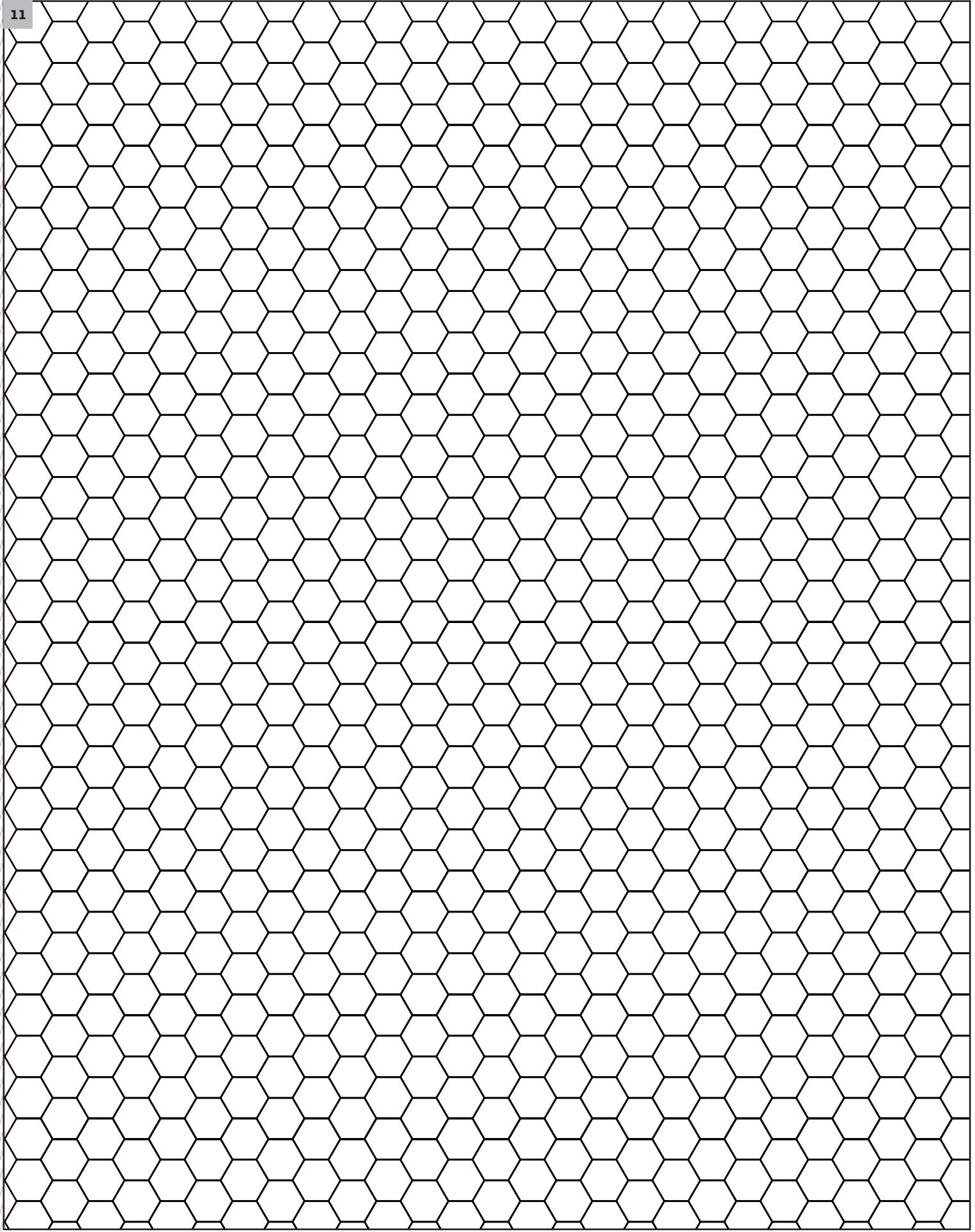
Engineers, mostly in spatial industry, astronomers but also architects, to build suspension bridges and stadiums. For instance, gardeners, lighting technicians and even computer graphics designers.



KEYWORDS FOR WEBSITES:

Conic - Ellipse - Parabola - Hyperbola - Envelope curves







2.Tiling a floor



Arts & Tiling



Kaleidoscopes



Where Am I?



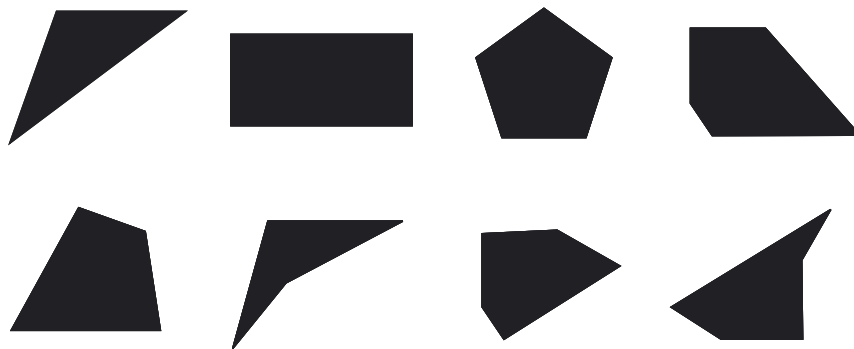
Do it yourself

MATERIAL: 1 sheet of shapes to cut out, 4 or 5 colour pencils or felt pens, 1 cutter or scissors

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Create the most beautiful tiling with one piece

Choose a shape and tile the plane with it. Make sure to leave no gaps or overlaps. Feel free to use colours. For each complete tiling, look at the 17 symmetry groups (listed page 8) and try to find which group it belongs to.



To notice

Is it possible to completely cover a floor without gaps or overlaps using any tile shape?

Many geometric or figurative shapes allow the tiling of the surface but not all of them. For instance, it is not possible with regular pentagons. In a regular tiling, you may observe a pattern that is periodically repeated by translation in two directions. Some tiling patterns may also have an axial symmetry or a rotational structure. Translations, rotations and symmetries can be classified in 17 distinct types of patterns. The study of symmetrical patterns is based on **Evariste Galois**' (1811-1832) "group theory", a French mathematician. Tiling patterns find application in mathematics, crystallography, in particle physics...

Do it yourself

MATERIAL: 1 sheet of paper with geometrical shapes to cut out, 4 or 5 colour pencils or felt pens, 1 cutter or scissors

20

21

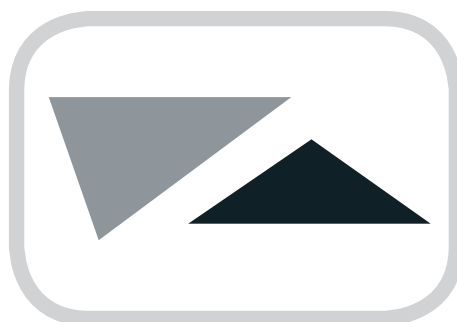
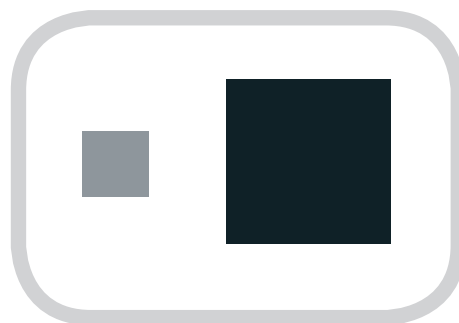
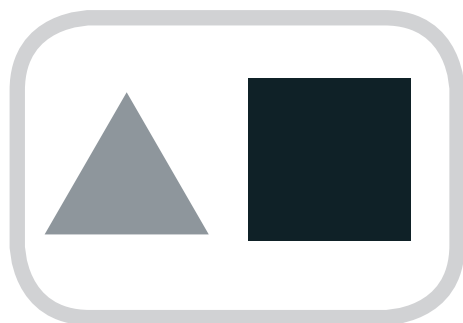
22

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Tiling with two shapes

- Choose a pair of shapes
- Tile the plane leaving neither gaps nor overlaps
- Is the tiling pattern regular (periodic)? If not, do you know why?

* Please, keep the congruence of the arcs of circles.





Do it yourself

MATERIAL: 3 geometric grids, 4 or 5 colour pencils or felt pens

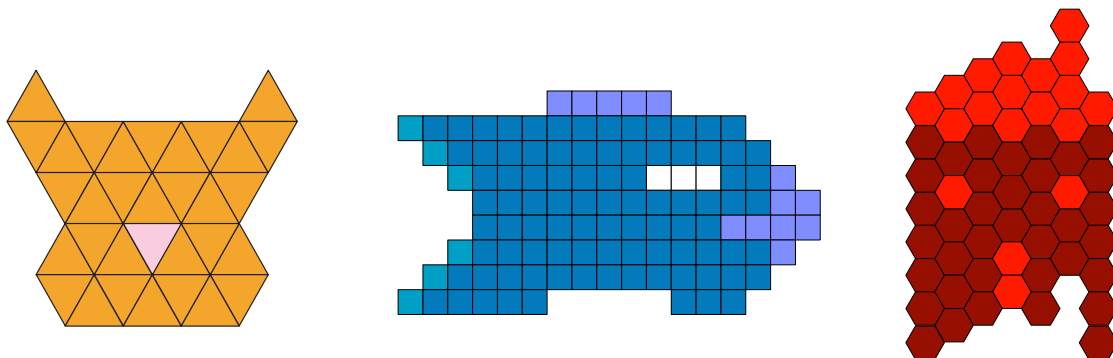
24

25

26

Cat, Fish, House.....

Turn a simple shape (triangle, square,...) into a figurative model who tiles the plane.
Feel free to use the grids provided for

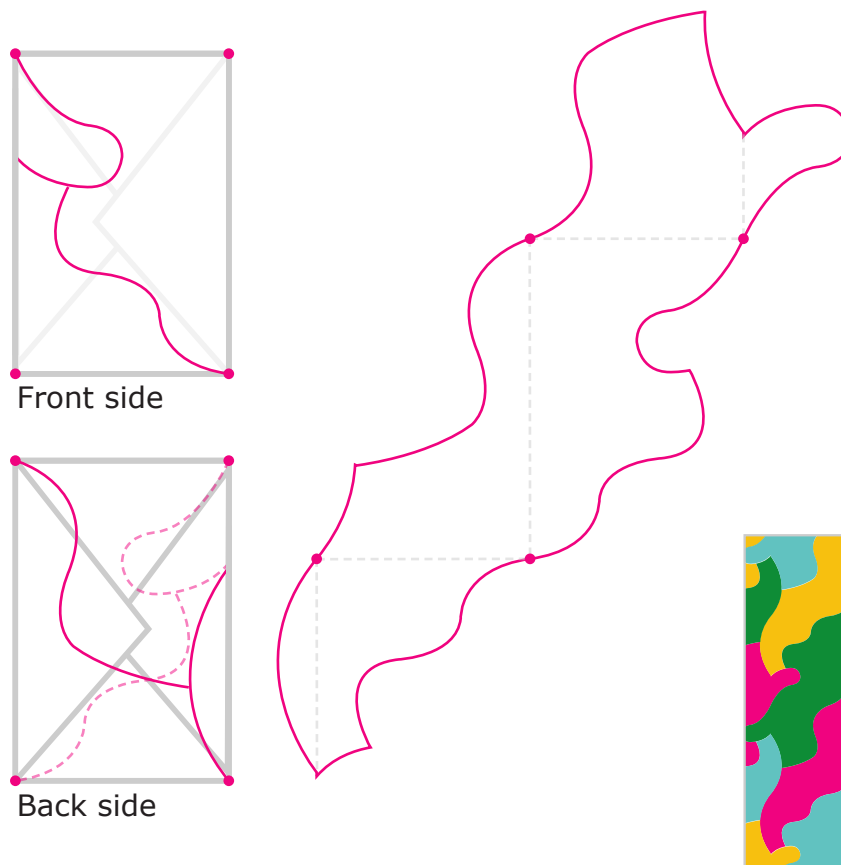


Do it yourself

MATERIAL: 1 closed envelope, 1 pencil, 1 cutter or scissors

The envelope technique

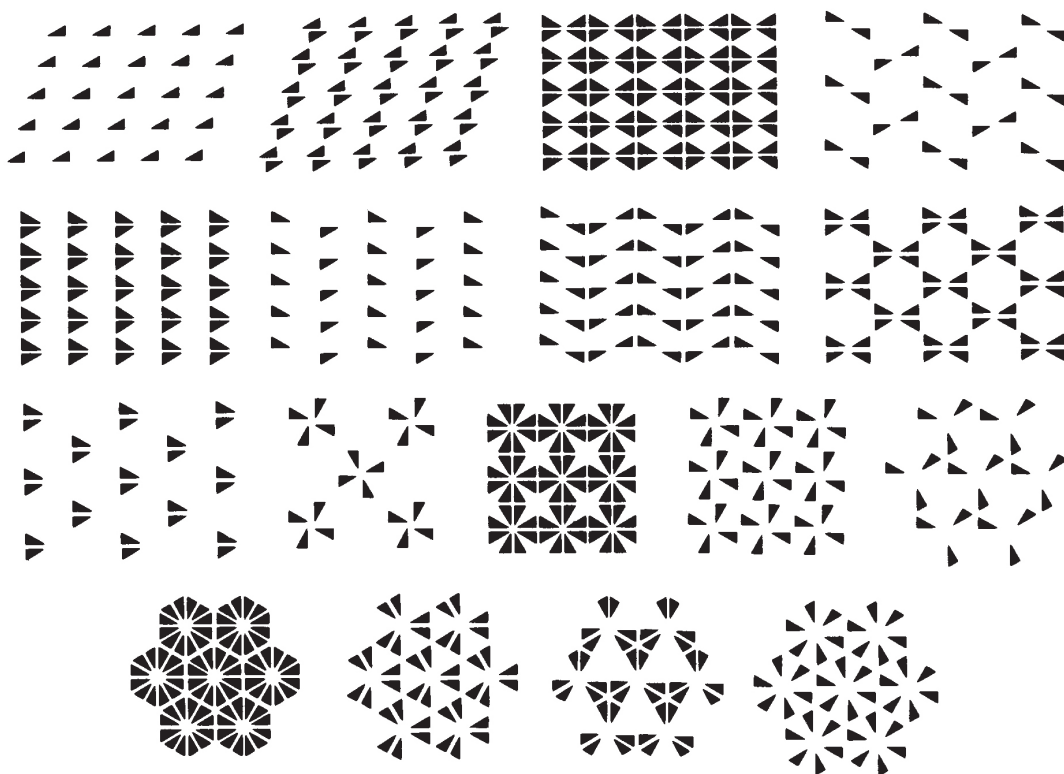
- Build up a rectangular (or square or triangular) envelope.
- Draw a line linking all the corners of the envelope. The line may go onto the other side of the envelope.
- Cut the envelope according to the line you have just drawn.
- Unfold the cut envelope and start tiling





Going further

The 17 groups of regular (periodic) tiling patterns



Maths-related jobs

Tiling is very often found in decoration, from wallpaper to floor tiles (kitchen, corridor, bath room...), but also in all kinds of fabrics (upholstery, African designs...).

KEYWORDS FOR WEBSITES:

Tiling - Tiling Groups - Evariste Galois - Escher

Kaleidoscopes



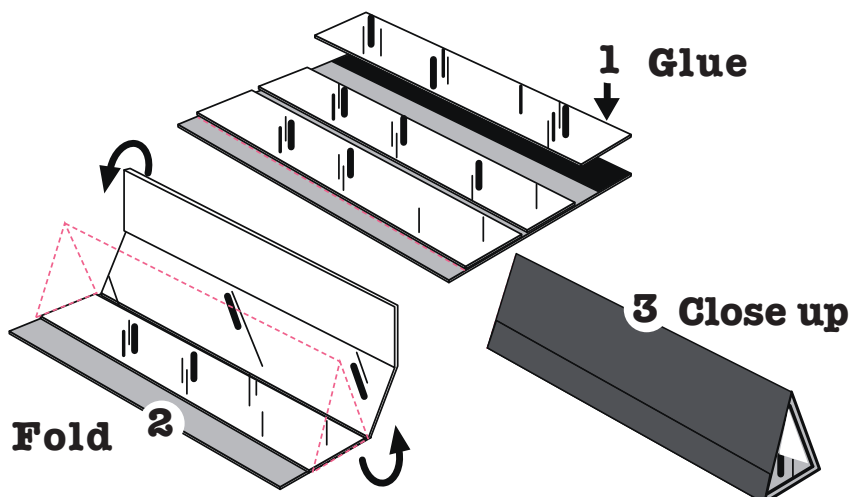
Do it yourself

MATERIAL: 2 kaleidoscope' models, 3 mirrors, 1 piece of cardboard, 1 glue stick, Adhesive tape

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Build two kaleidoscopes

Build two kaleidoscopes: one with an equilateral triangle base, and a second one with an isosceles right triangle



Do it yourself

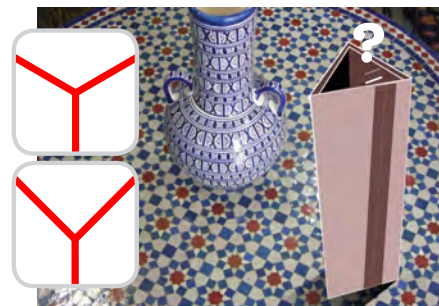
MATERIAL: Mosaic models to be observed, 2 kaleidoscopes

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Observe the symmetries in the kaleidoscope

- Choose a mosaic and place the appropriate kaleidoscope on it.
- You may observe the mosaic, bigger.
- Place one of the mirrors on the red lines.
- You will find the tiling pattern of a house floor.



To notice

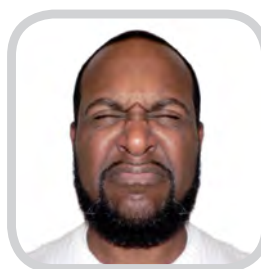
Periodic tiling patterns are motives repeated indefinitely. Here, the motif is reproduced with mirror symmetries to create a square, and hexagon or any other polygon with an even number of sides. The same technique was used by artists like **Escher** with more figurative motives.

Do it yourself

MATERIAL: 1 picture of a face, 1 cutter or scissors

Who is behind the mirror?

- Take the picture of a face.
- Cut it into symmetric halves and create mirror effects (as shown here).
- Where is the right face?**



Kaleidoscopes



Do it yourself

MATERIAL: 3 pyramid patterns, 1 cutter or scissors, 3 triangle mirrors, 1 glue stick

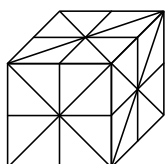
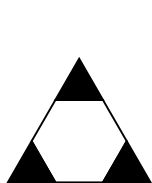
30

31

32

Mirror pyramids

- Create a mirror pyramid with one of the provided models.
- Place objects or a liquid of your choice at the bottom of the kaleidoscope.
- What do you see?



Going further

If you cut a regular polyhedron (cube, tetrahedron...) following all the symmetries, you obtain pyramids. If you turn them into kaleidoscopes, they enable to find the original polyhedron, but also a whole family of volumes with the same base symmetries.

Maths-related jobs

- Craftsmen creating zeliges mosaic (Fez - Morocco)
- Technician designing a new tiling pattern (Kenitra - Morocco)



MOTS CLÉS POUR SITES WEB:

Kaleidoscope - Mosaic tiles - Tiling - Symmetries - Escher

Where Am I?



Do it yourself

MATERIAL: 1 polystyrene sphere or 1 ball, 1 felt pen

Sphere tiling

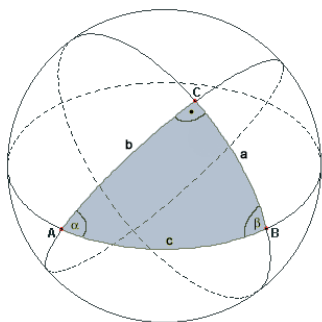
Observe the tiling on the sphere. Can you imagine other motives?

Try to draw them on the ball.

For each tiling pattern, calculate the angles of the tiling polygon.

To Remember

As for the plane, the tiling of a sphere is done without gaps or overlaps using one or more spherical polygons (which can be applied onto the sphere). The tiling is regular if only one regular shape (equilateral triangle, square...) is used evenly around each vertex. A regular tiling on a sphere is the spherical deformation of a regular polyhedron. For instance, tiling a sphere helps defining the optimum number of satellites to cover any spot on Earth.



Going further

Calculating the angles of polygons drawn on a sphere, you can discover properties of spherical geometry.

- What is the sum of the angles of a spherical triangle?
- Can you build a spherical triangle with 3 right angles?
- What is the relation between the sum of the angles of a spherical polygon and its area?



The bear hunter story

This tale is about a hunter who goes after a bear.

He walks straight ahead towards the South for an hour and suddenly notices that the bear turned towards the East. The hunter follows the trail and walks for another hour to reach a point where the bear turned again, towards the North this time. Again, the hunter follows the tracks for yet another hour when suddenly, to his surprise, he realizes that he is back where he started!

Question 1: What colour is the bear?

Question 2: How many solutions are there?

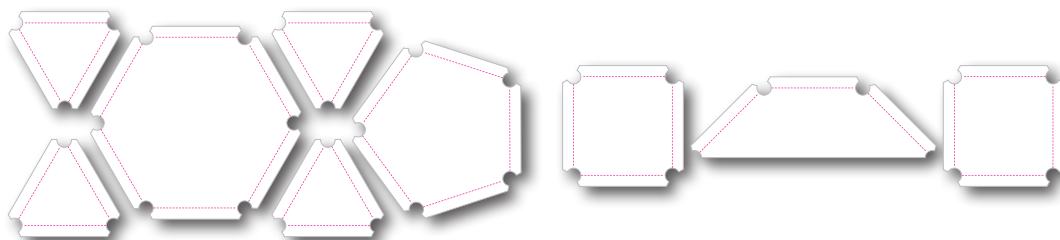
Do it yourself

MATERIAL: 1 sheet of patterns for regular polygons, 2 or 3 soft cardboards, rubber bands

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Build a cardboard ball

Use these regular polygons to build a sphere-like ball.

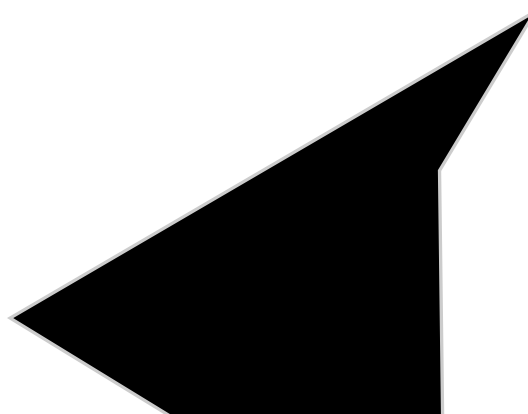
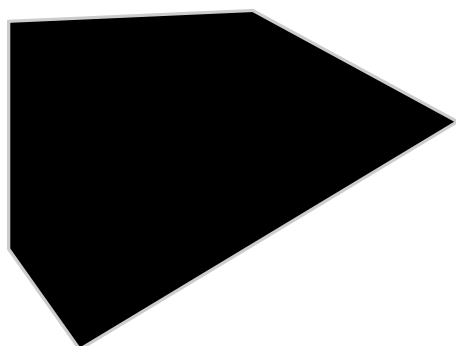
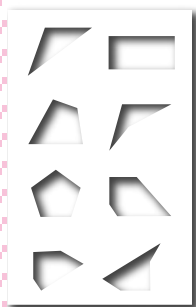
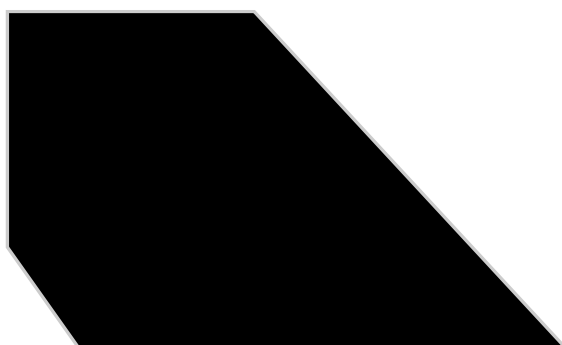
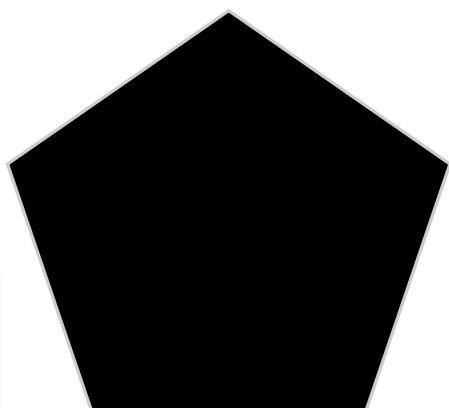
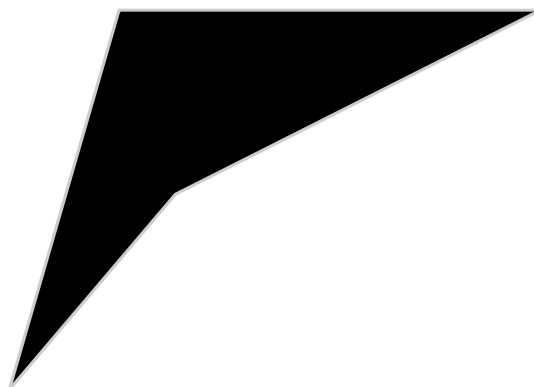
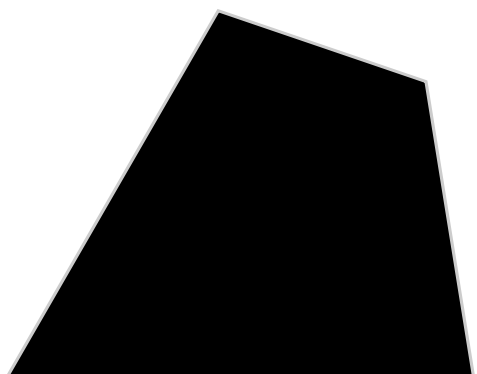
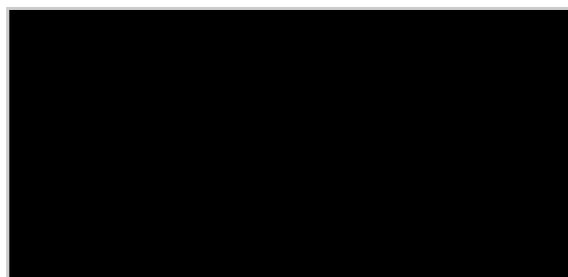
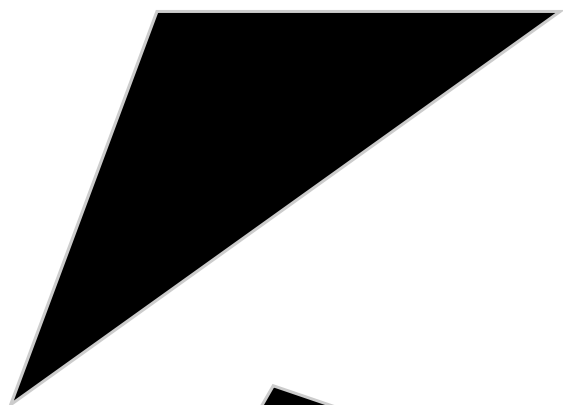


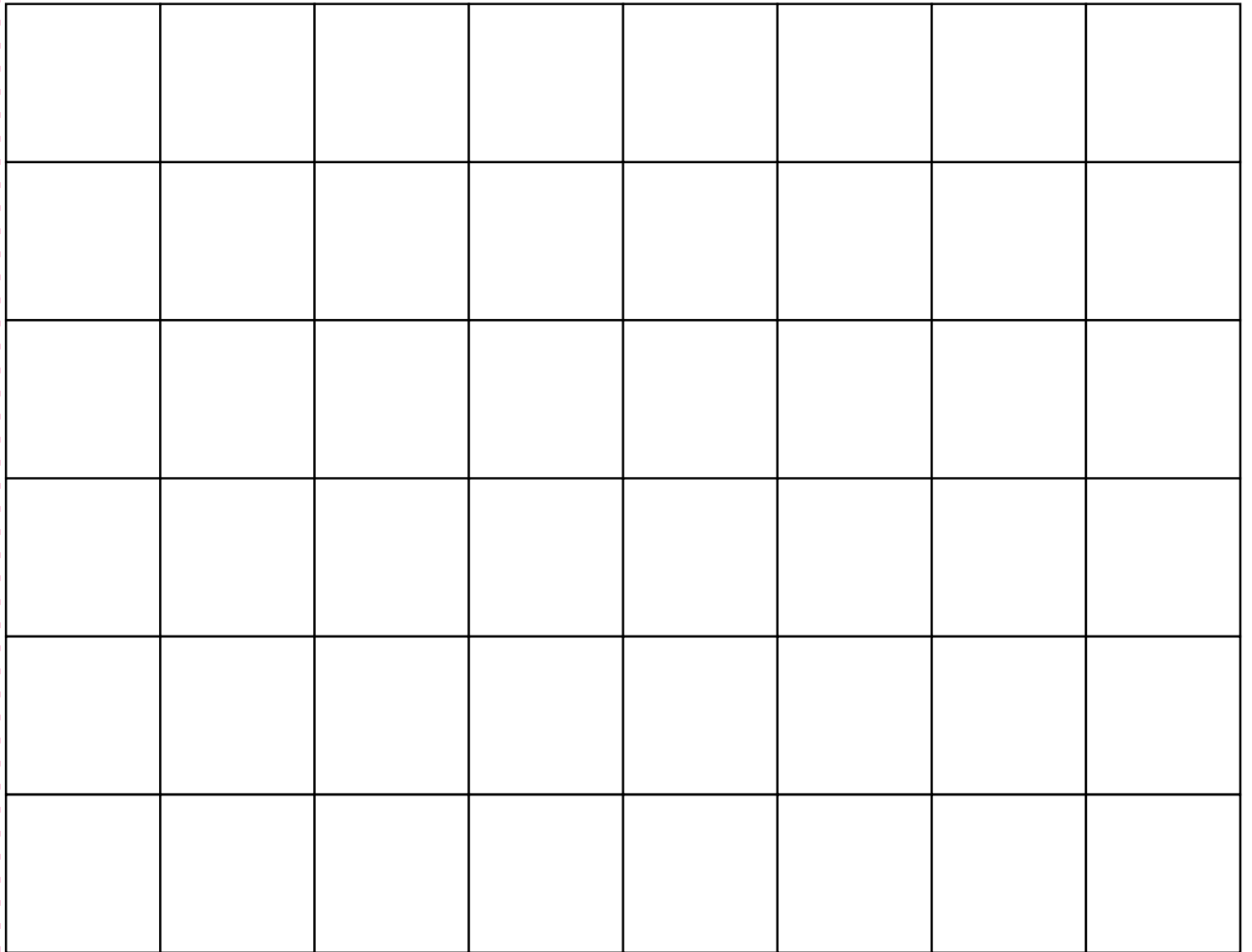
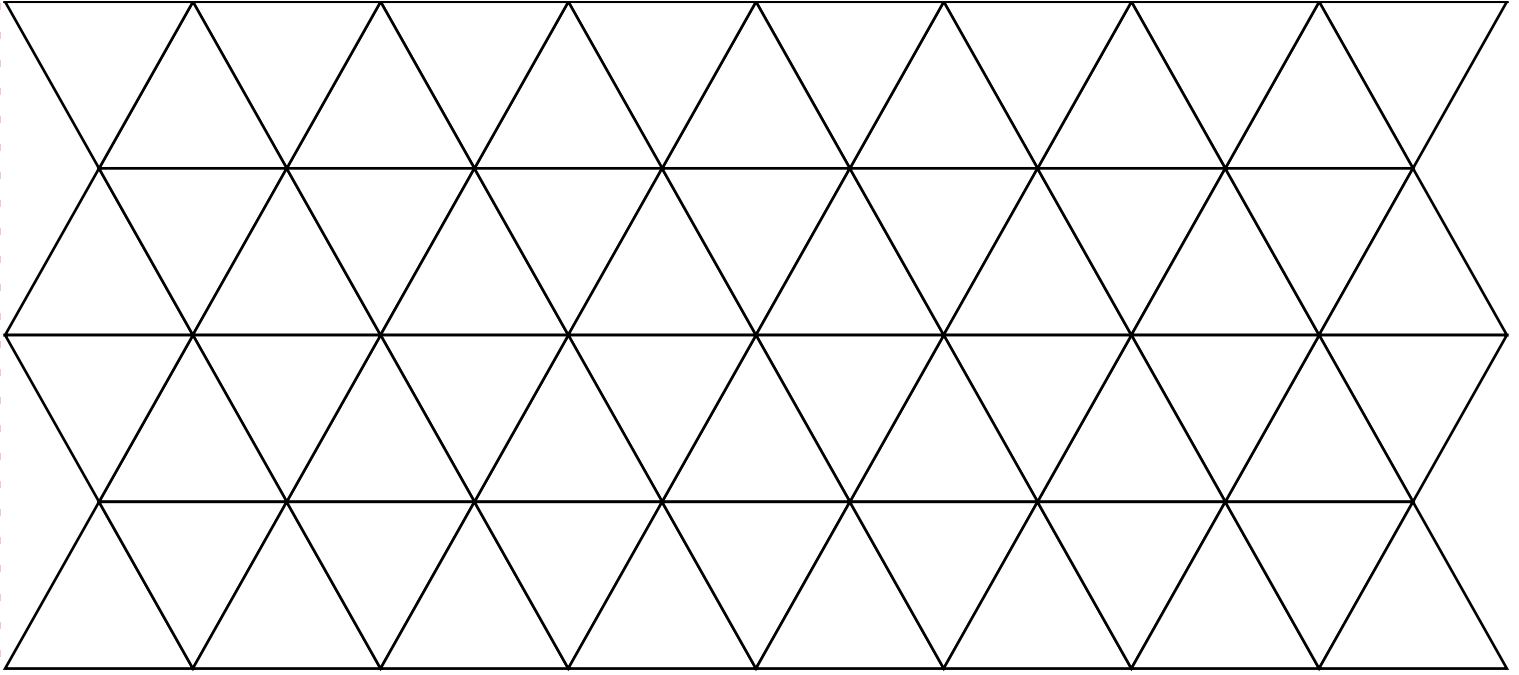
Maths-related jobs

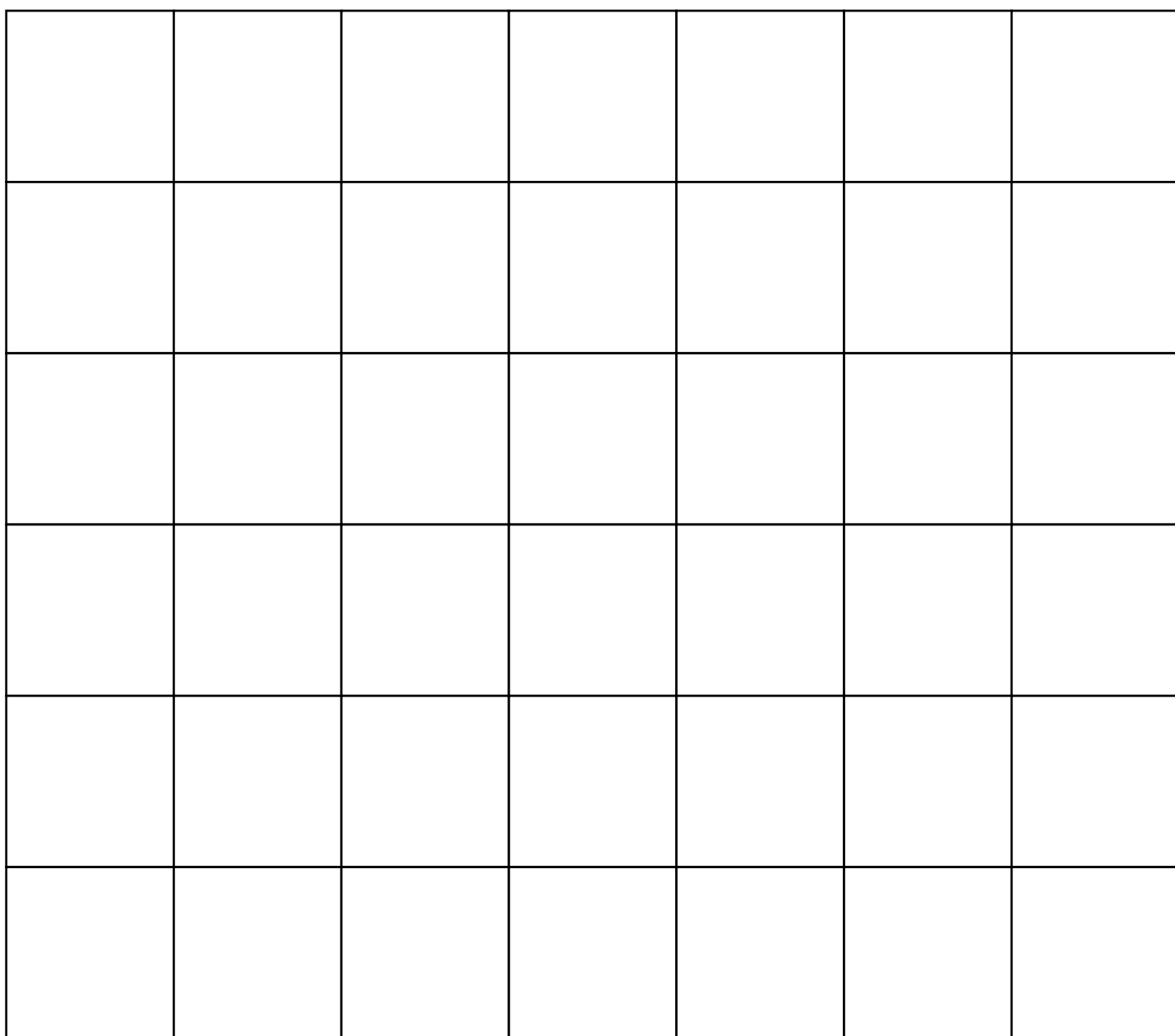
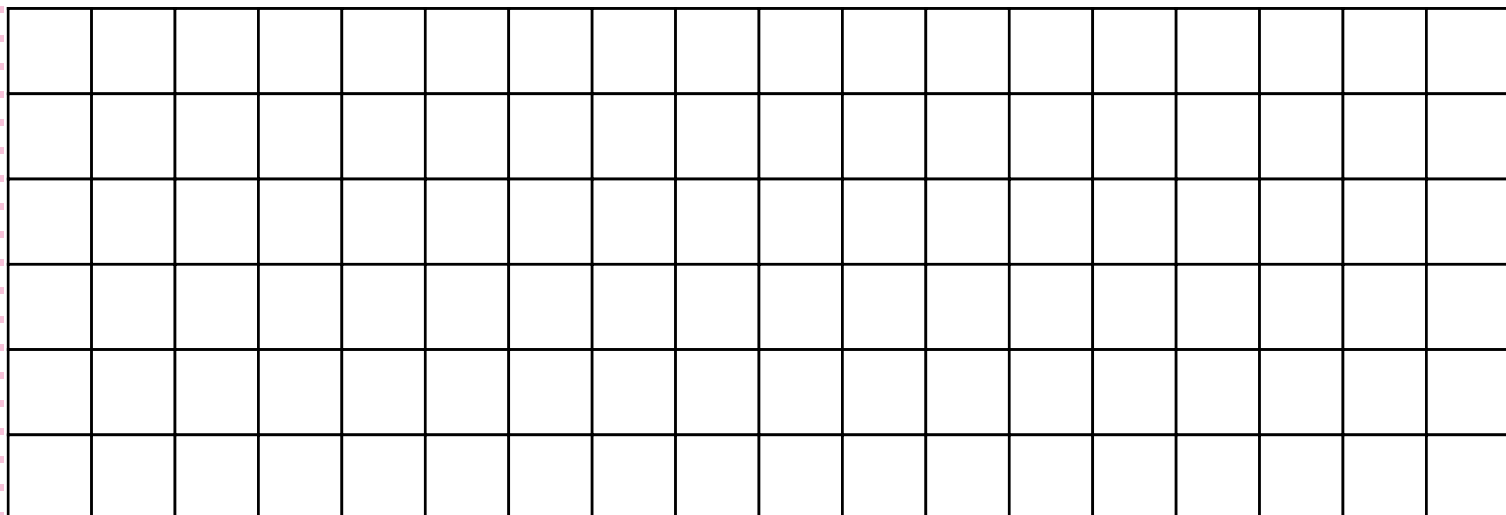
Architects take inspiration from spherical structures. Covering the Earth with as little satellites as possible is a direct application of spherical geometry for satellite communication and positioning systems (GPS or Galileo).

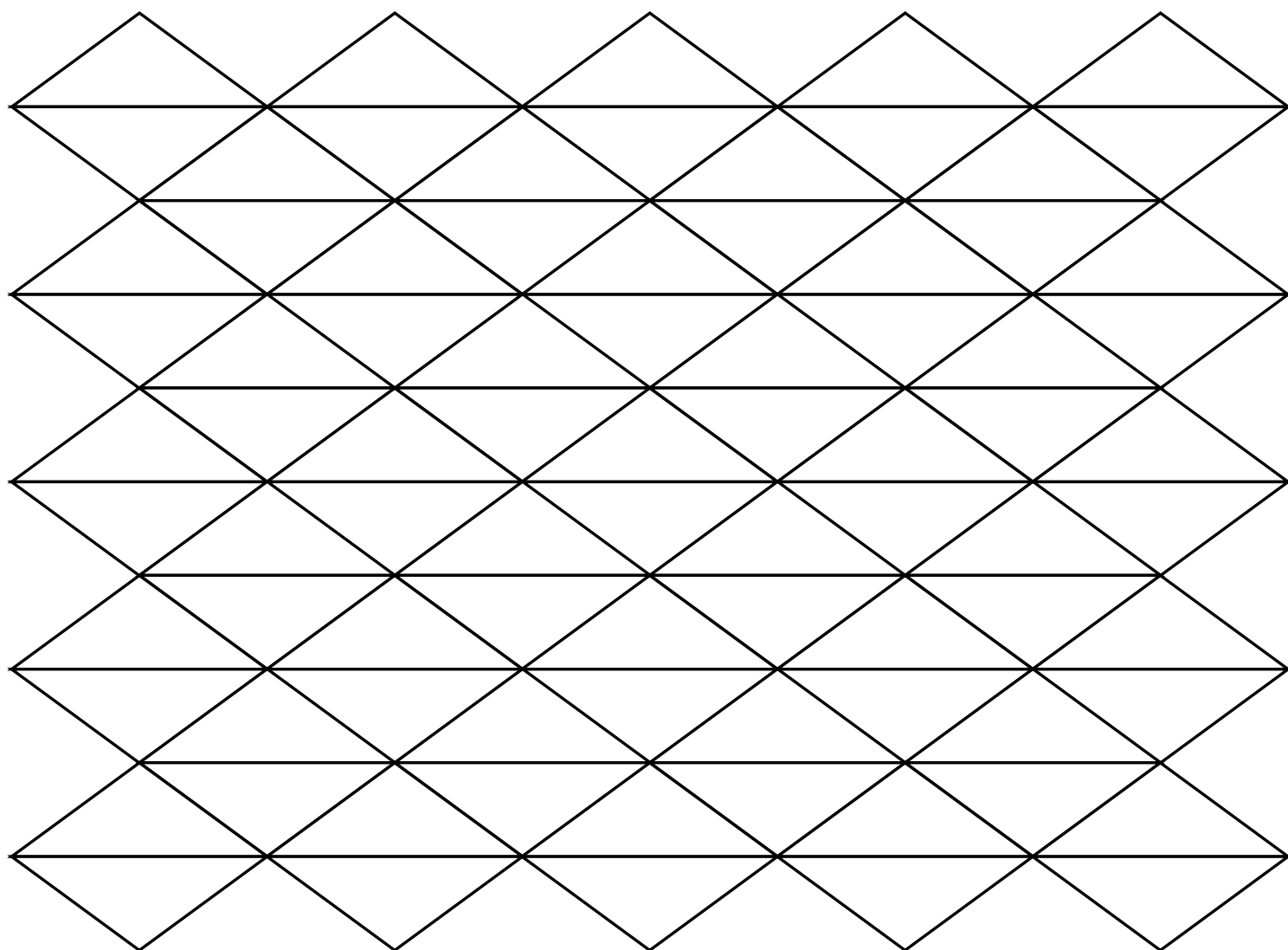
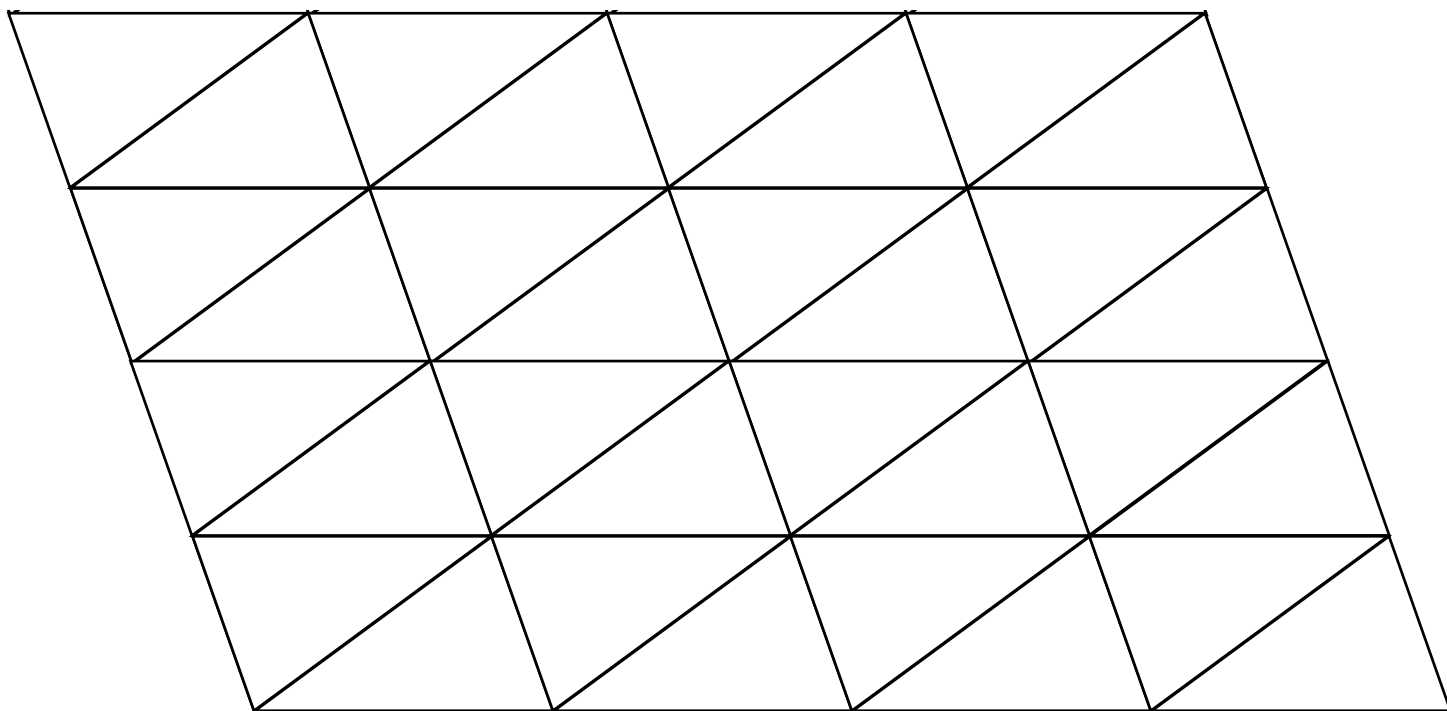
KEYWORDS FOR WEBSITES:

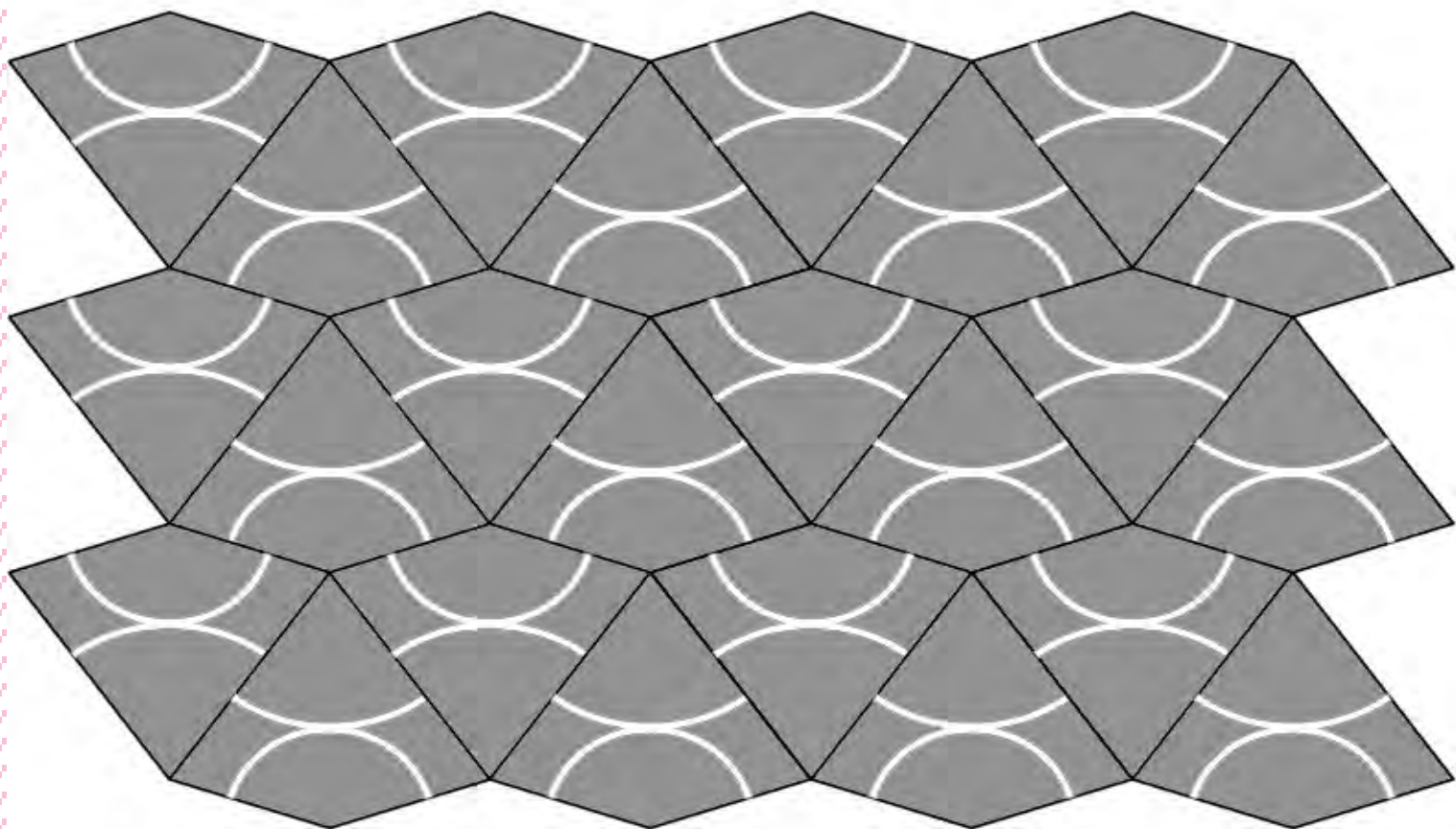
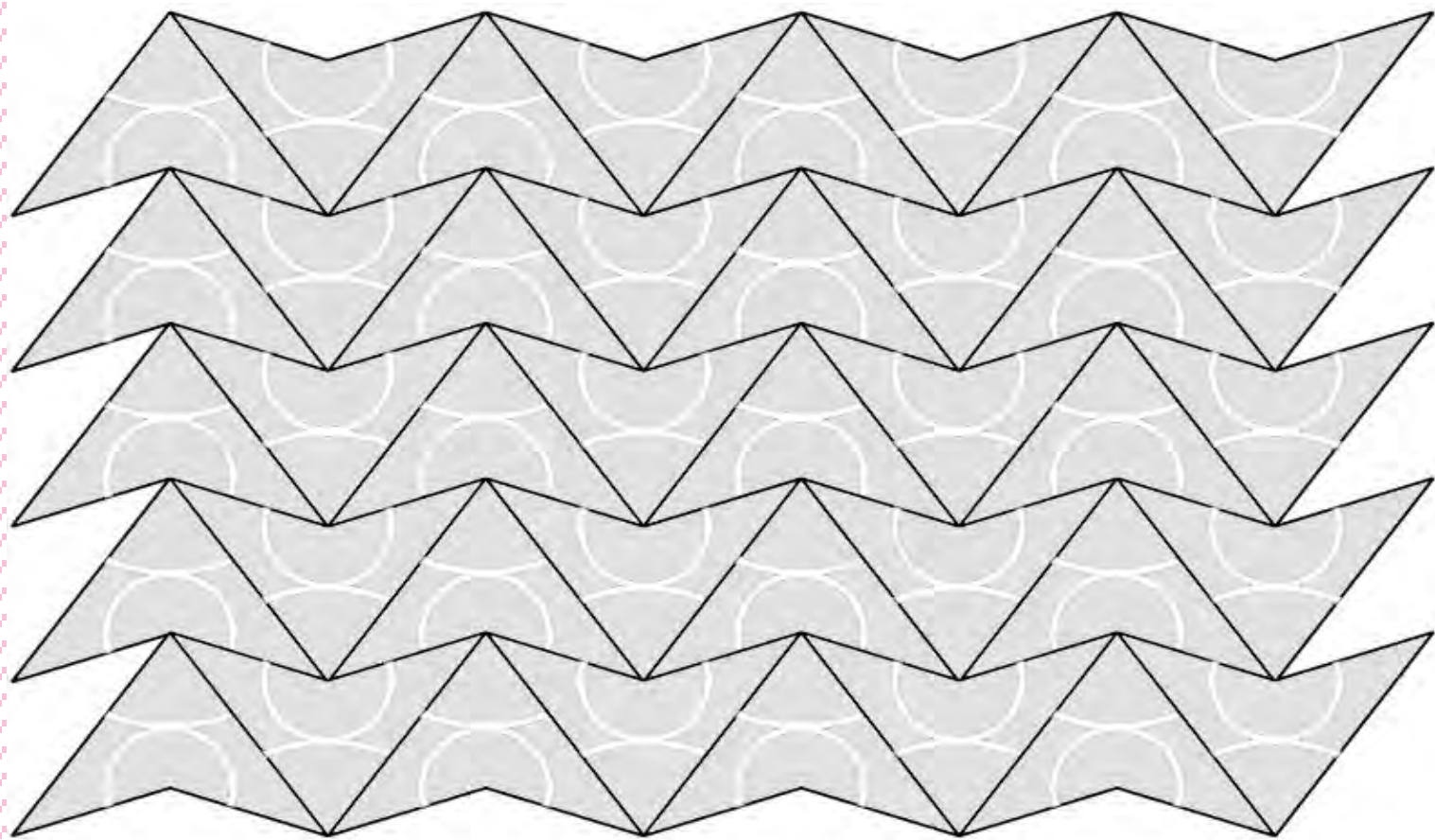
Spherical geometry - Sphere tiling - Escher

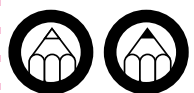


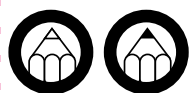




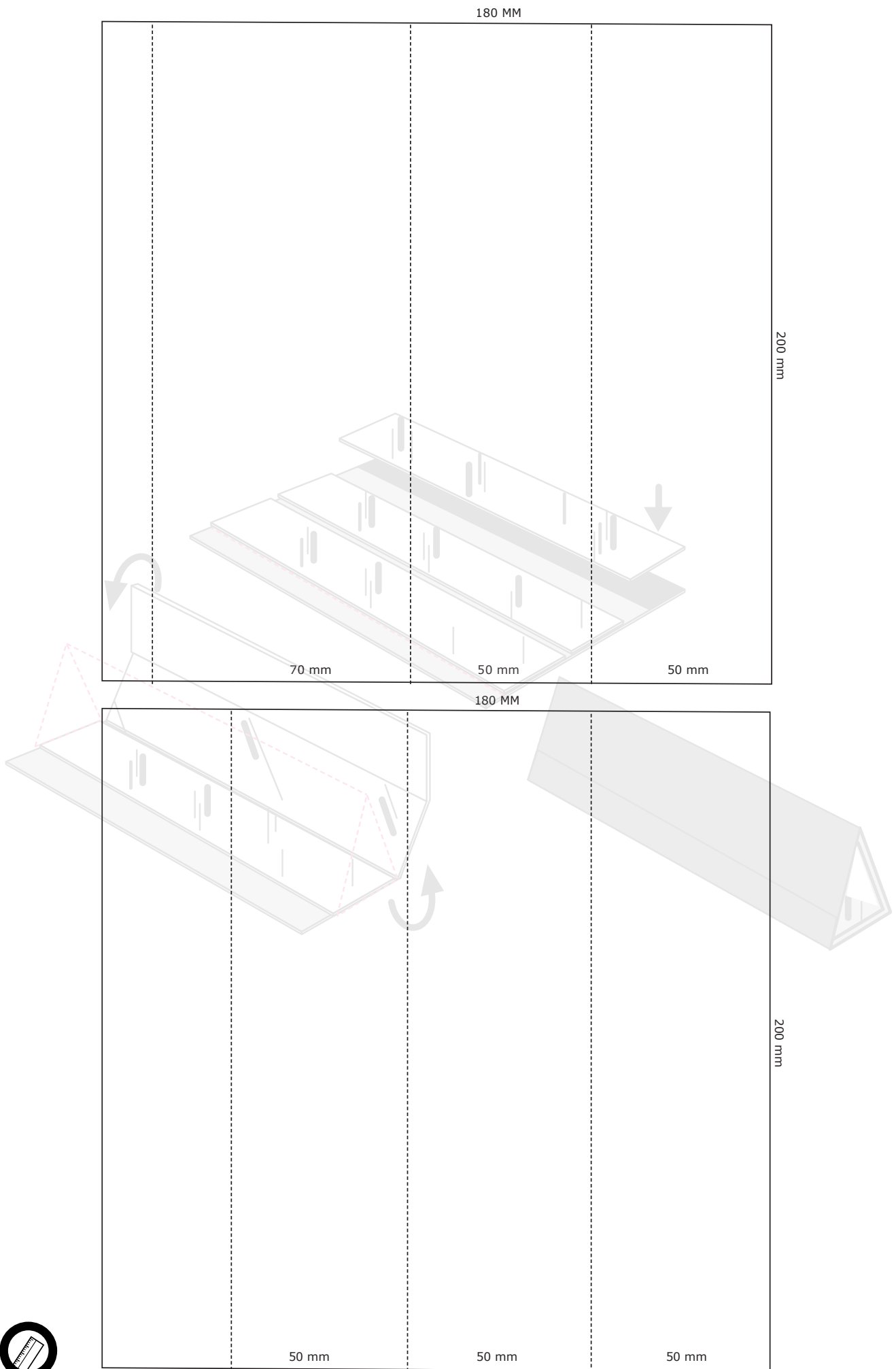


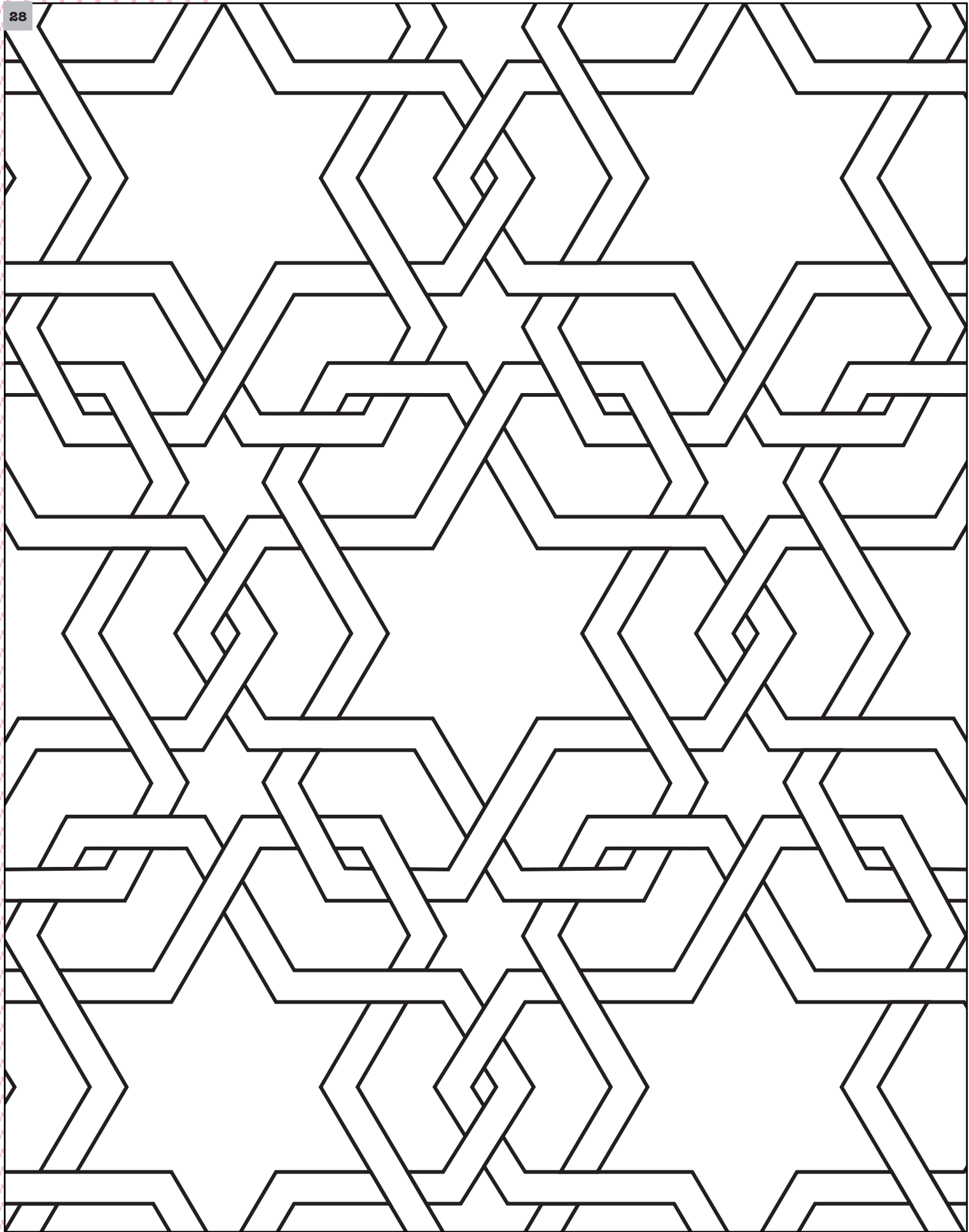


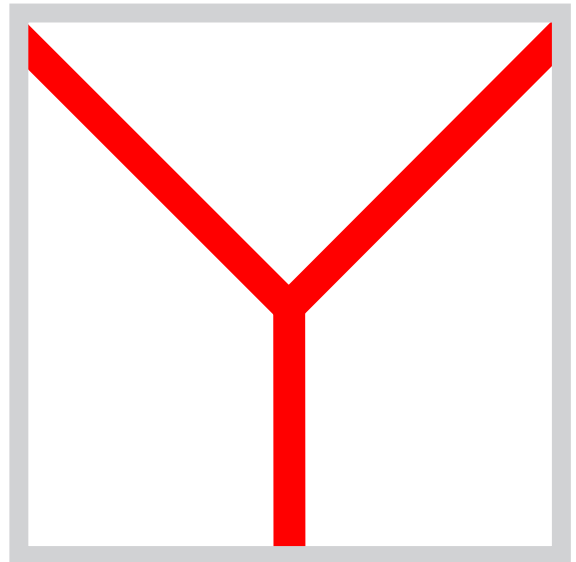
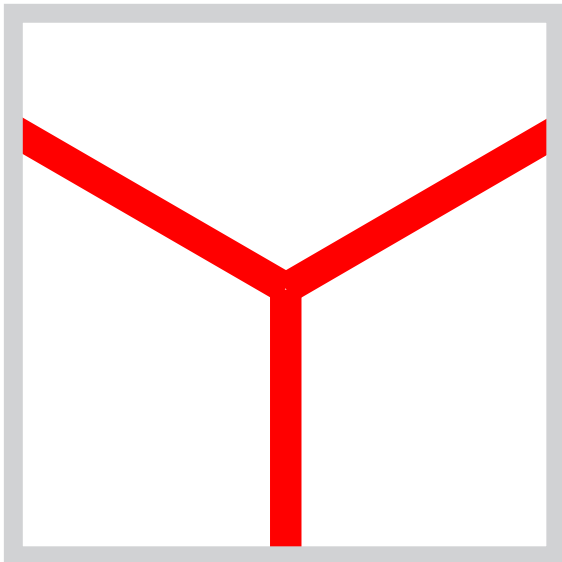




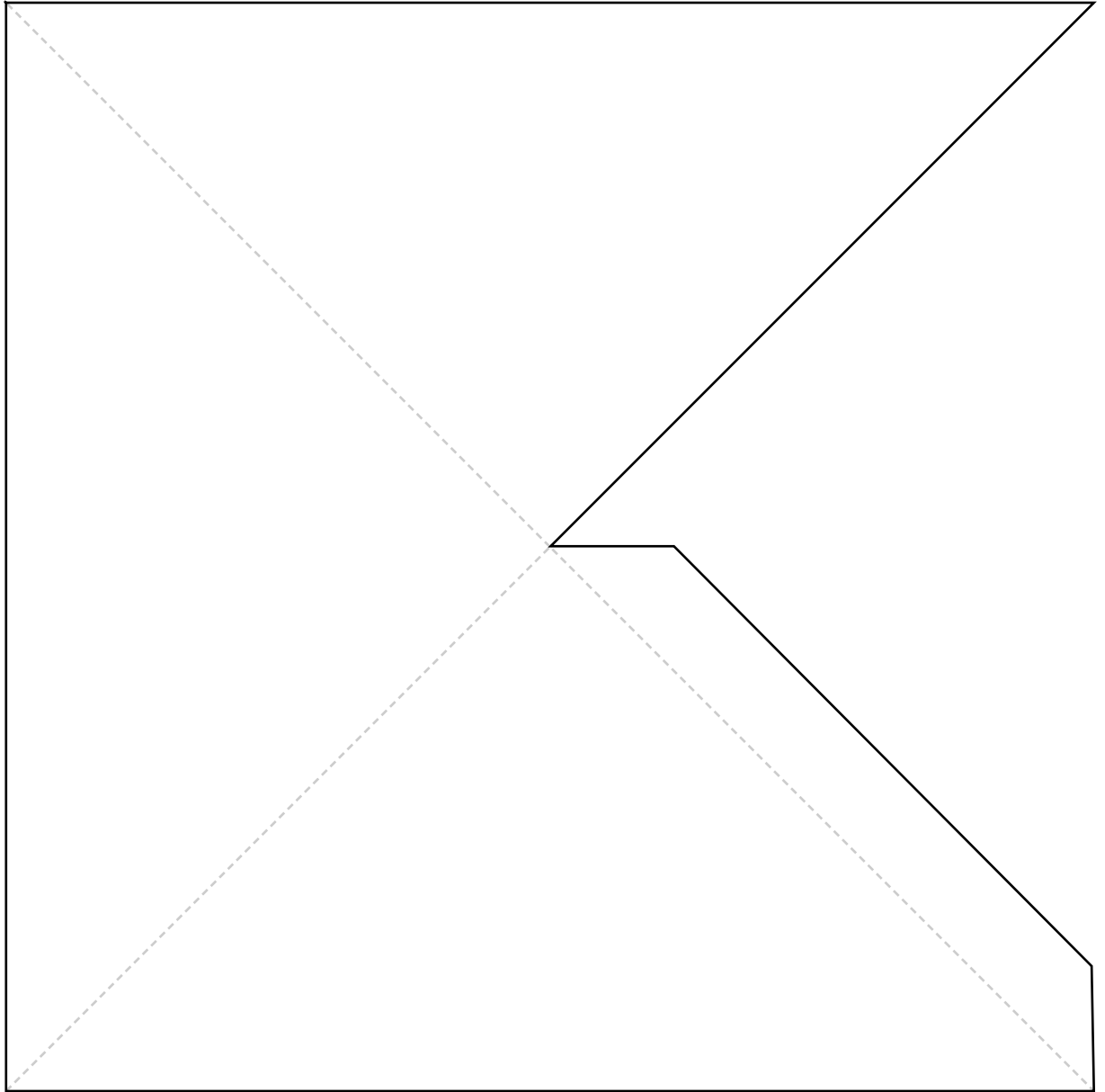




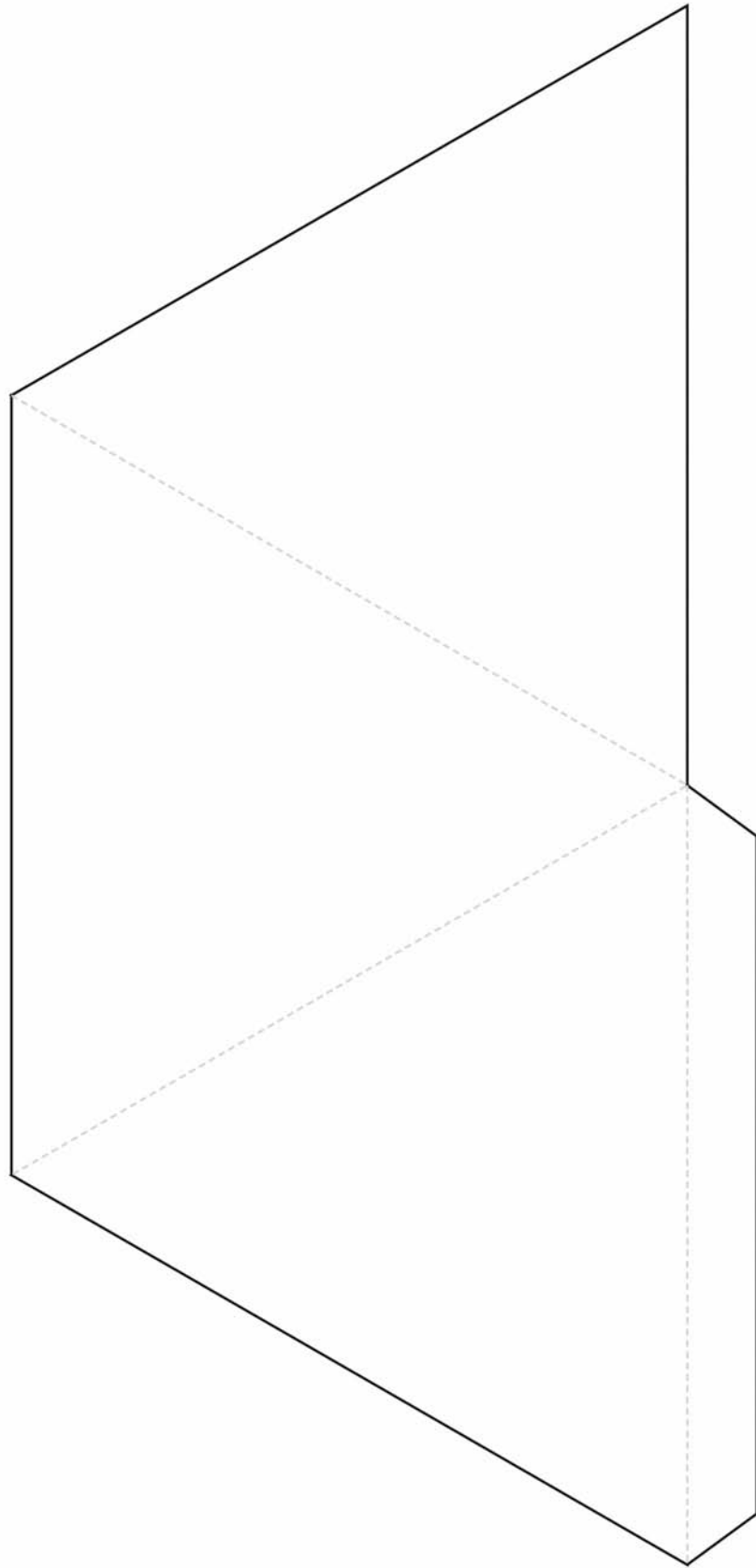




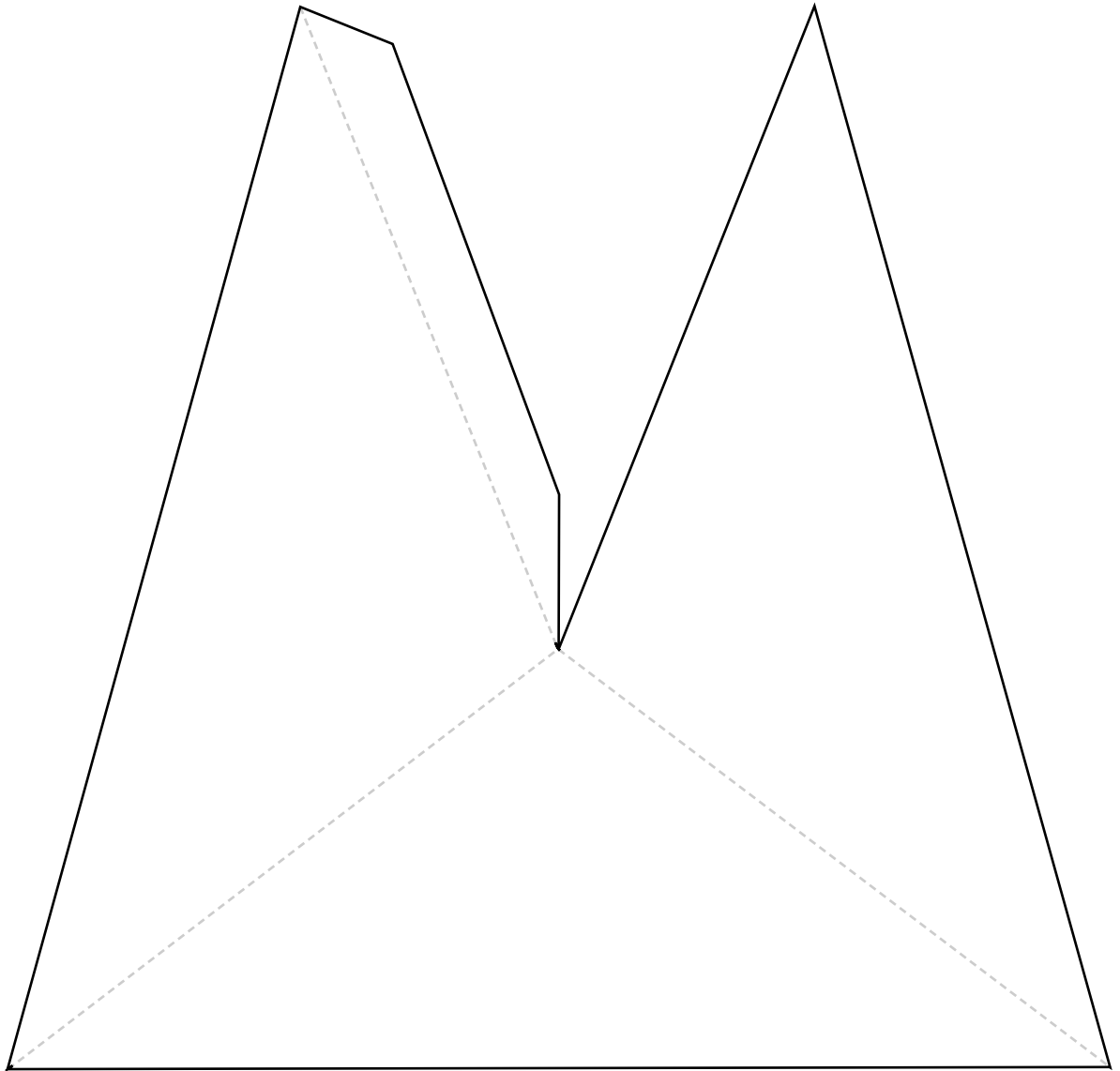
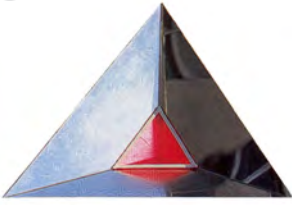
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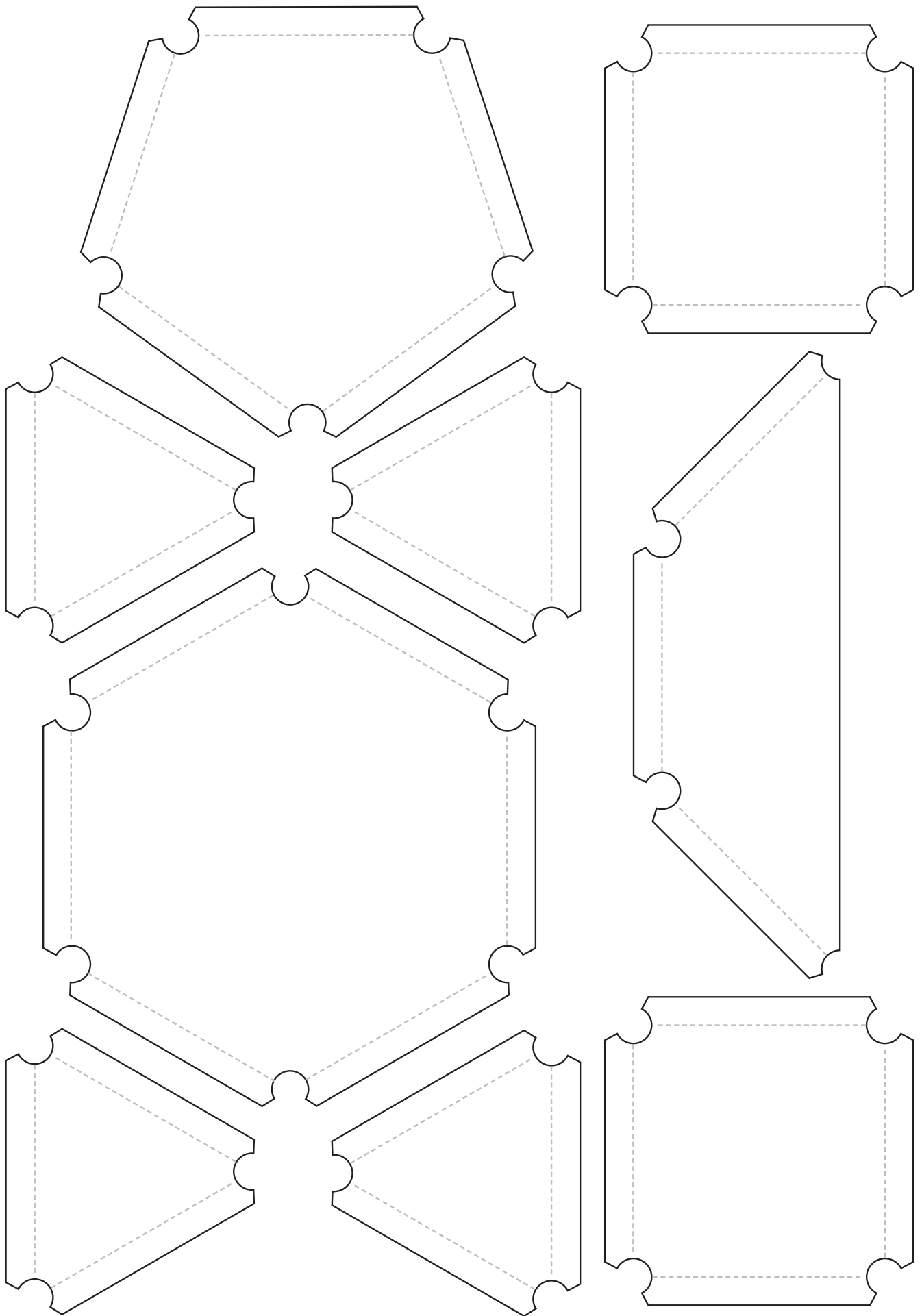


2



3







3. Filling space



Piling Oranges!



Polyedrons



Complex problems

Piling Oranges!



Do it yourself

MATERIAL: 1 squared sheet, change coins, marbles or oranges

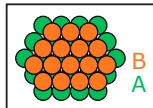
39

Keep piling...

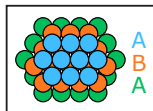
- Place a maximum of circular units in a square of 1 unit side, of 2 units, 3, 4, ... units side.
 - Try to pile up a maximum of marbles on a square base of 10 units.
- Calculate the density for each pile.



To notice



You can place more than 100 disks in a square of 10 units' side!
At which level is the number of units higher than the square of the side?
In the plane, the maximum density* that can be obtained for identical disks is 90.6%. There is less than 10% of empty space.

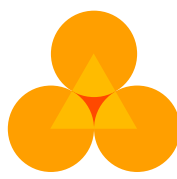
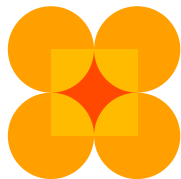


In three dimensions, when the sphere packing is regular or irregular, the maximum density can be obtained (like crystal lattices) when the spheres are at the vertices and centres of the faces of cubes regularly packed in the space. This type of packing is called "faces-centred cubic packing". Its density is 74%.

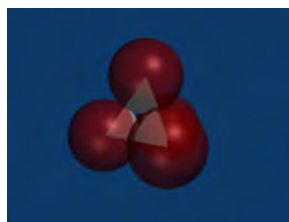
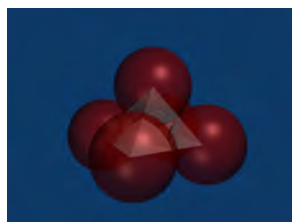
For the case of irregular packings with flat or different size marbles the problem has not been solved, yet.

**Density is proportional to the volume (or the surface) of the marbles in the pile inside the pyramidal or cubic container (or the square).*

Going further



Calculating the packing density of the disks is the same as comparing the area of the disks inside the square (or triangle) with the area of the square itself (or triangle)..



Calculating the density of the balls is the same as comparing the volume occupied by the balls within the pyramid (or tetrahedron) with the volume of the pyramid itself (or tetrahedron).

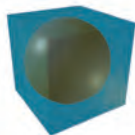
Piling Oranges!



Going further

1st question:

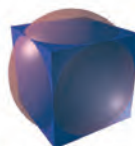
Between a kilo of ground coffee and a kilo of coffee beans, which of the two will occupy a lesser volume?



2nd question:

Place a sphere inside a cube in such a way that it touches the 6 faces of the cube. Give an estimate of the ratio of the volumes (without any calculation).

Estimate their surface ratio. Calculate to check that **these two ratios are equal**. This is one of the methods used by **Archimedes** to find the area and volume of a sphere.



3rd question:

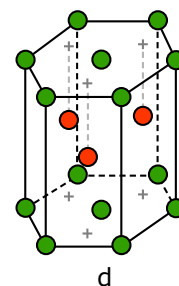
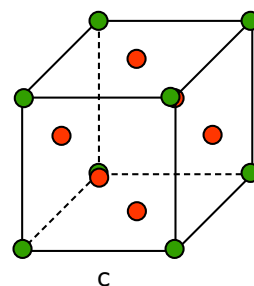
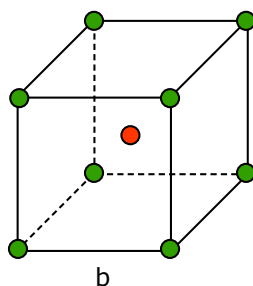
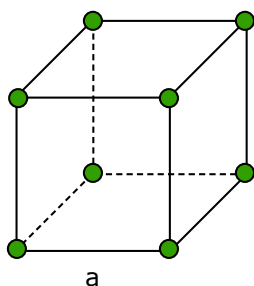
Inside a cube, place a sphere in such a way that it is tangent to the edges of the cube. Ask the same questions again.

This time the comment is as follows:

The volume ratio is equal to twice the maximum density of the piles of spheres.

Some results are easy to find:

- a. Simple cubic packing - Density: $\pi/6$
- b. Centred cubic packing - Density: $\pi\sqrt{3}/8$
- c. Faces-centred cubic packing - Density: $\pi\sqrt{2}/6$
- c. Faces-centred cubic packing - Density: $\pi\sqrt{2}/6$



Maths-related jobs

Any company interested in packing objects, seeds, pills... Physicists, engineers who are interested in material and atomic packing. Packings are also used for the encryption of messages and their automatic corrections (Hamming codes).

KEYWORDS FOR WEBSITES:

Packing - Density - Kepler - Hale

3. Filling space

Polyhedrons



Do it yourself

MATERIAL: Polyhedron patterns, scissors or cutter, glue

40

41

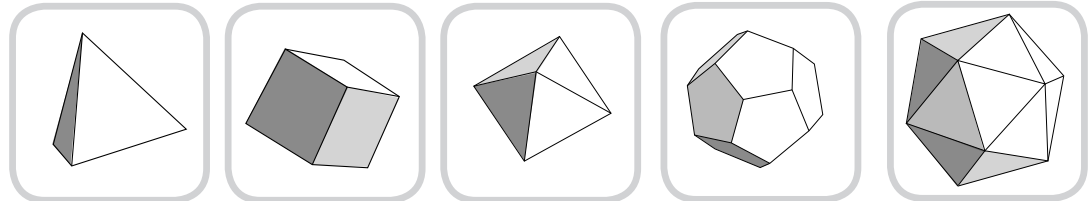
42

43

44

Make your own polyhedrons

- From a cardboard pattern or assembled regular polygons, build the 5 Platonic solids, a double tetrahedron, a square-based pyramid...
- Choose a polyhedron and turn it.
- How many Faces does it have? How many Summits? Vertices?



To notice

A regular polyhedron is a solid whose faces are made of one regular polygon repeated and connected evenly around each summit.

There are 5, called **Plato's polyhedra**.

It is semiregular if the faces are made of 2 or 3 types of regular polygons.

There are 13 of them called **Archimedean polyhedra**.

Whether regular or not, there is a relation between the number of vertices, edges and faces of convex polyhedra:

$$F + S = V + 2$$

This relation was discovered by **Euler** in 1752.

What happens in the case of a planar graph?

In the case of a solid with a hole in it? With two holes?

Fill-in the table below:

Polyhedron	Faces	Summits	Vertices
cube	6	8	12
...			

Is there any relation between these numbers?

Do it yourself

MATERIAL: Polygon patterns, cardboard sheets, scissors, a hole-puncher or a pastry-cutter

45

46

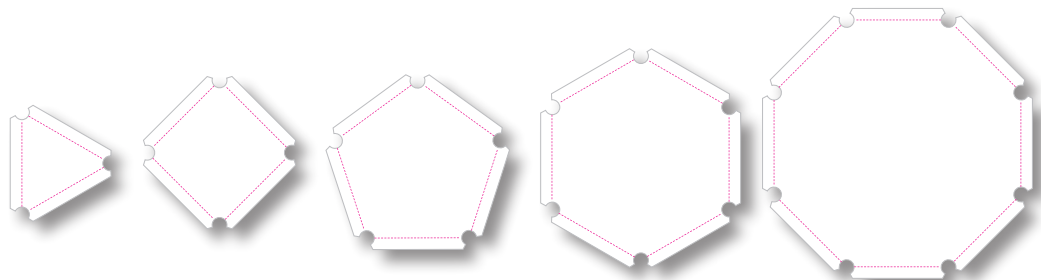
47

48

49

Build and conclude!

- Cut out each group of regular polygons with 3, 4, 5, 6 or 8 sides.
- Shape the cut-outs into a regular polyhedron, and then, into a semiregular.
- How many different regular polyhedra can you make? How many semiregular polyhedra?



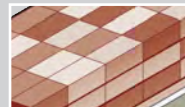
Maths-related jobs

These spacial structures are used by architects. You may also see them in nature. Physicists are very much interested in them (packings), but also biologists and naturalists with the diatoms for instance.

KEYWORDS FOR WEBSITES:

Polyhedra - Euler - Plato - Archimedes

Complex problems

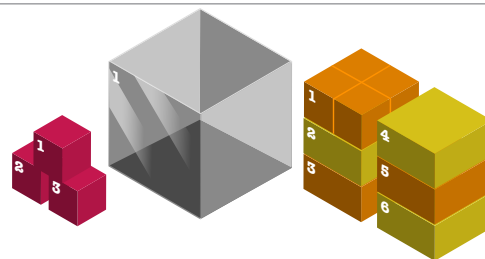


Do it yourself

MATERIAL: 3 little wooden cubes (or foam), 6 square boards made of 4 assembled cube blocks of the same material, 1 3x3x3 cubic box

Fill the box!

The challenge is to try to fill the box using all the blocks.



To notice

We often have to face this problem in our everyday life: how to fit as many objects in a box as possible or a maximum of boxes in the trunk of a car. This is a **complex problem** for mathematicians, in particular, as the more objects, the longer it takes to find a solution. Moreover, the time spent on the problem increases exponentially.

Do it yourself

MATERIAL: 1 rucksack, objects to put inside

The best rucksack

The Challenge is to try to fill the rucksack with the maximum possible value keeping in mind that the **total weight must not exceed 15kg**.



To notice

A pyramid twice as tall is eight times larger in volume than the small pyramid. Reorganizing, one can compare the volumes of these two types of pyramids. With height three times that of the small pyramid, one can even find the formula for the volume of any pyramid:

$$\text{Volume} = \frac{\text{Base} \times \text{Height}}{3}$$

With other solids, the packing problem is generally much more difficult!

To go further

Cubes can fill up space without gaps or distortions. Can you think of other examples of polyhedra with the same property?

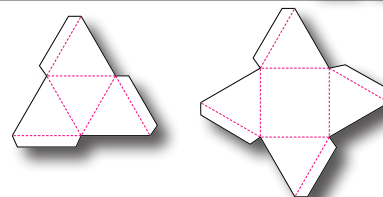
Do it yourself

MATERIAL: 2 tetrahedron patterns, scissors or 1 cutter, glue

50 51

The great Pyramid

Construct **6 regular tetrahedra** and **6 square-based pyramids** using the same triangular faces. With each set of solids, build a large pyramid (or tetrahedron) twice as tall as a single pyramid (or tetrahedron). Compare the volumes of the two pyramids (or tetrahedra).

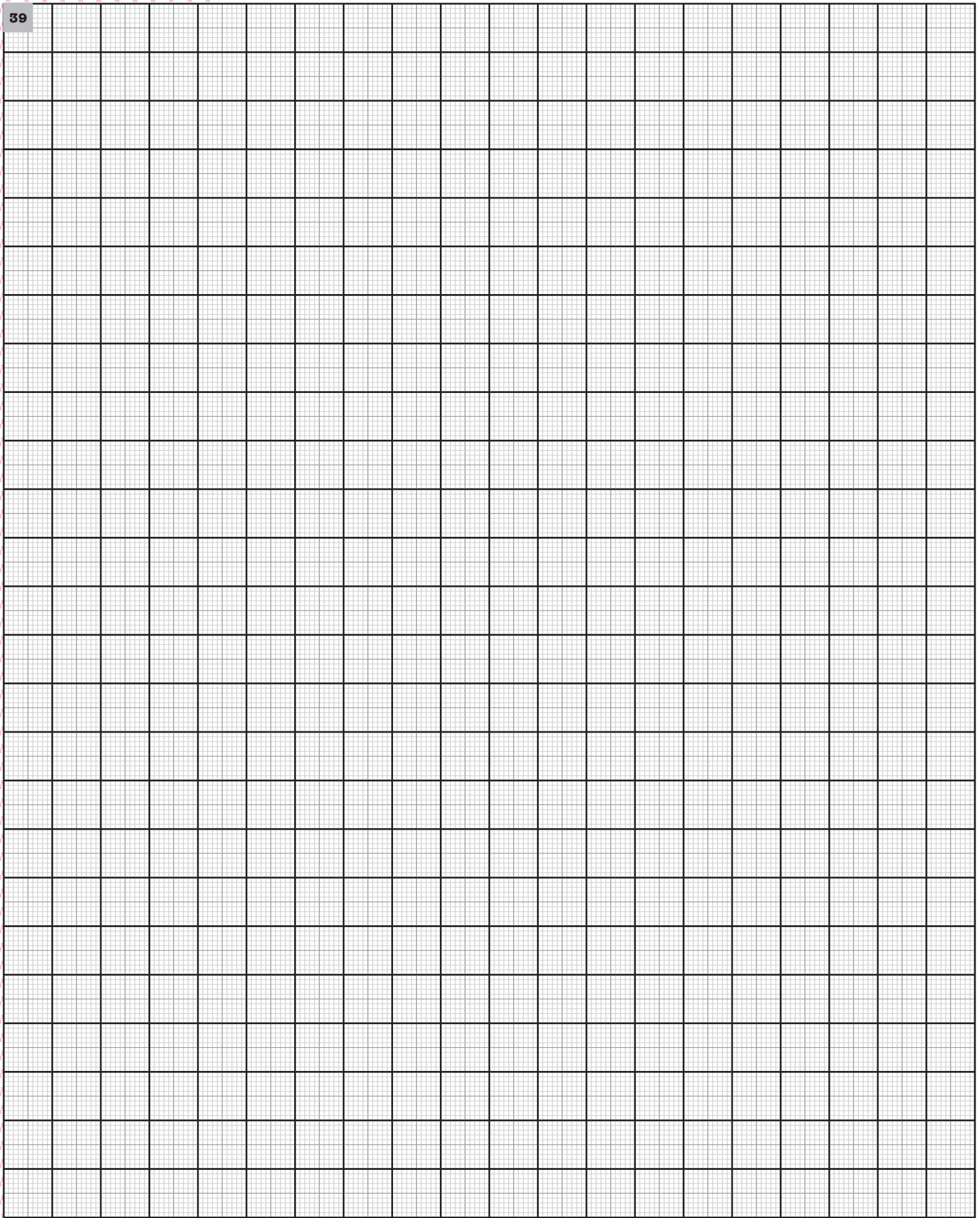


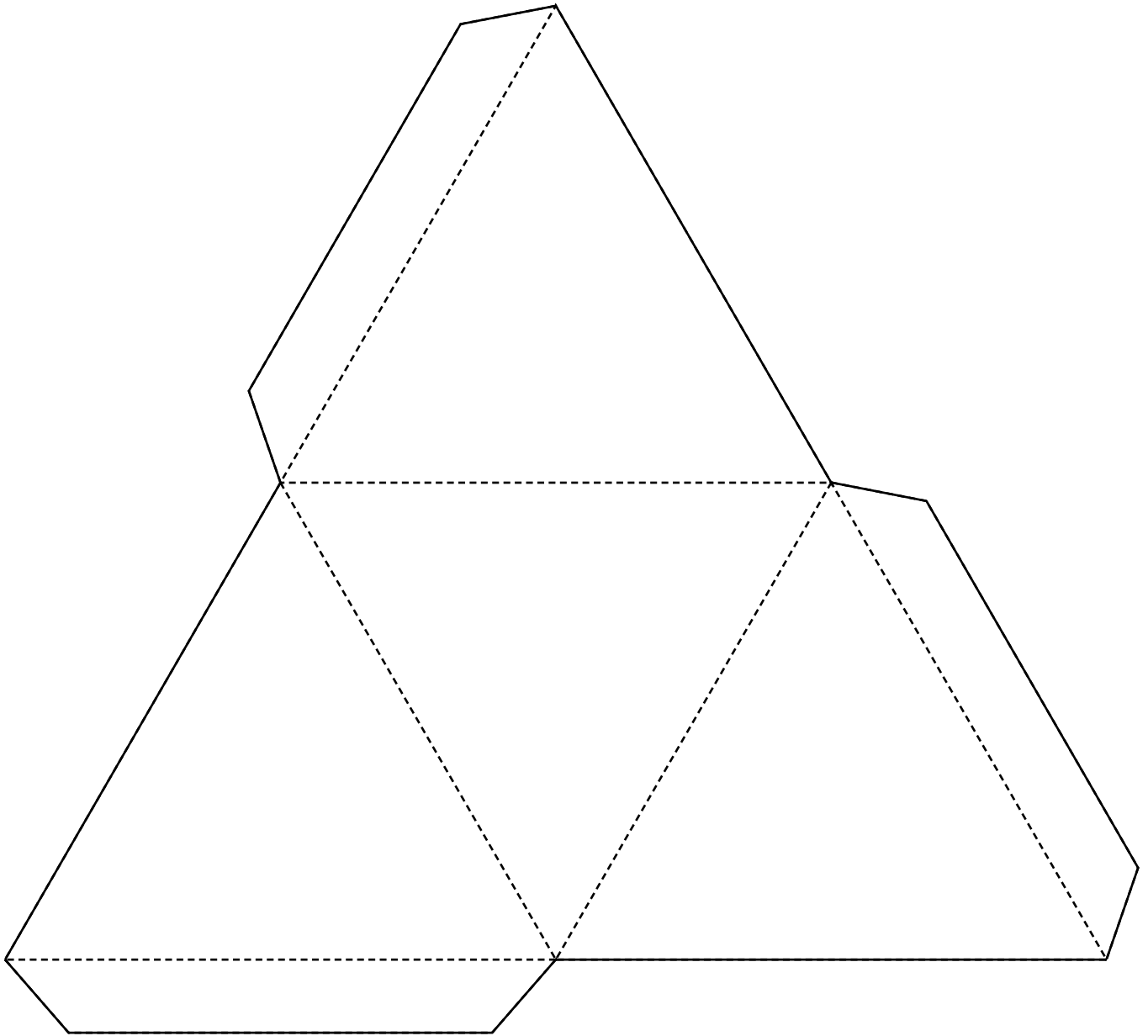
Maths-related jobs

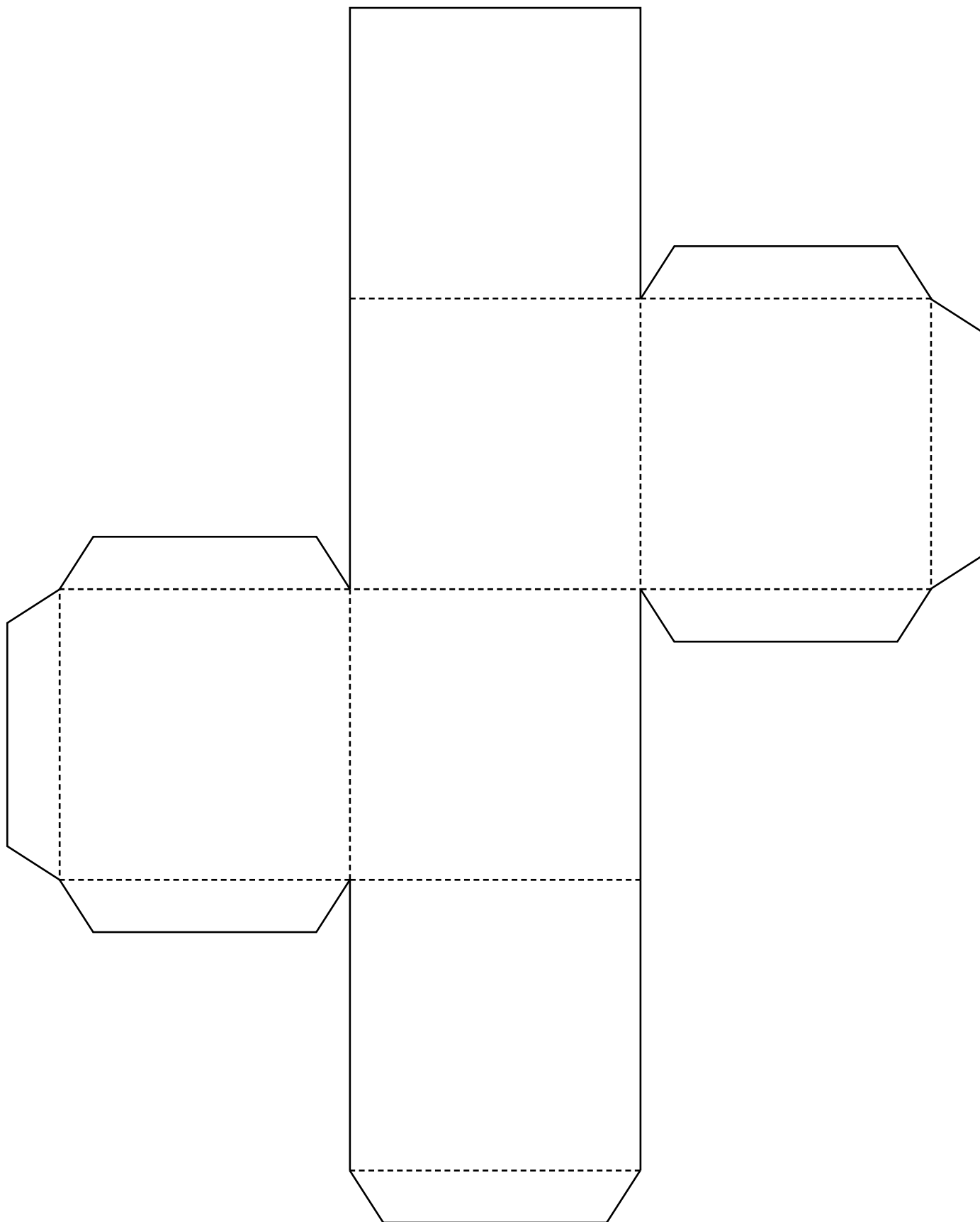
Beside our everyday life, not only heavy transport drivers but also air, maritime carriers and any packaging company in general have to face this problem on a daily basis.

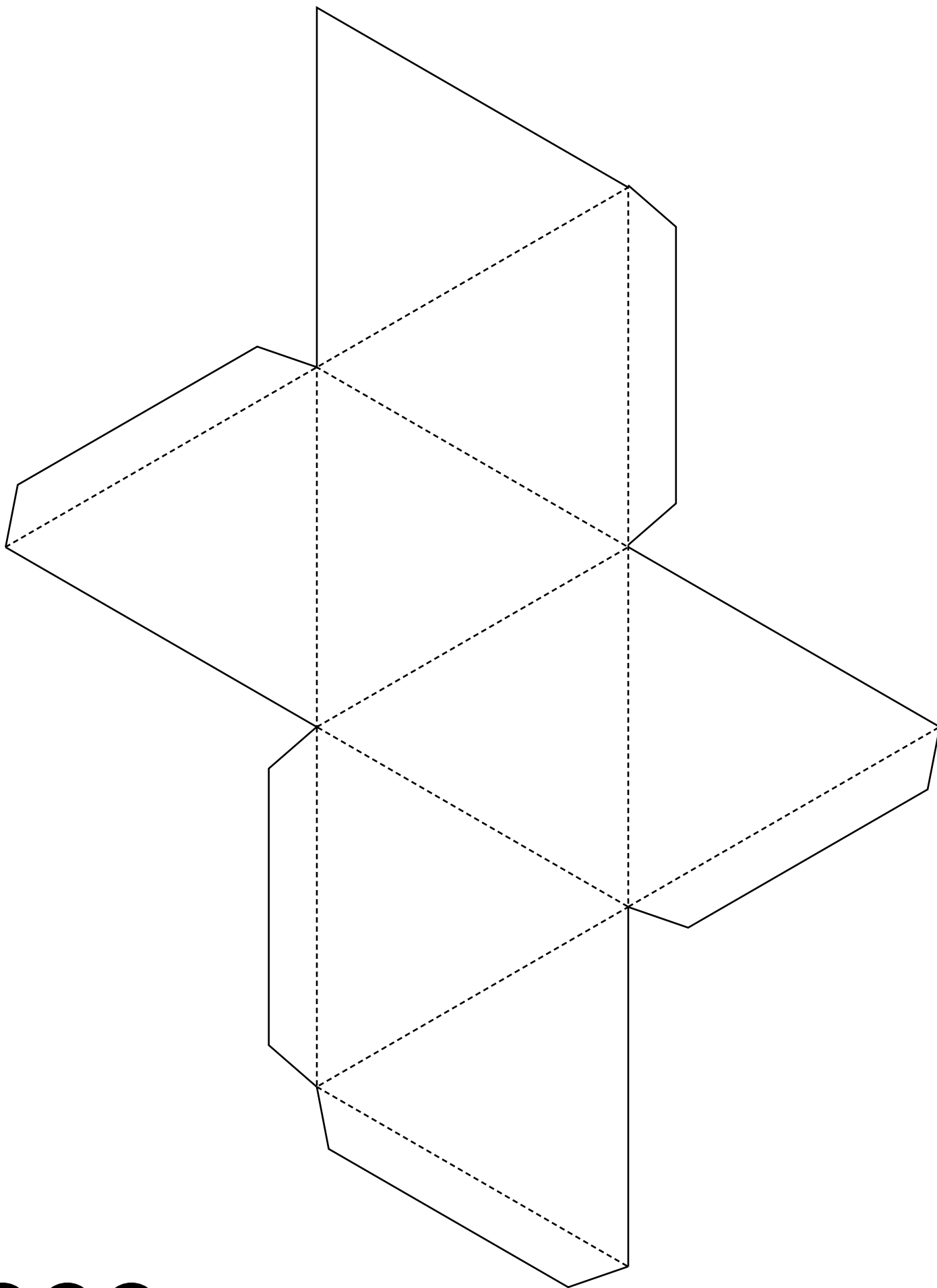
KEYWORDS FOR WEBSITES:

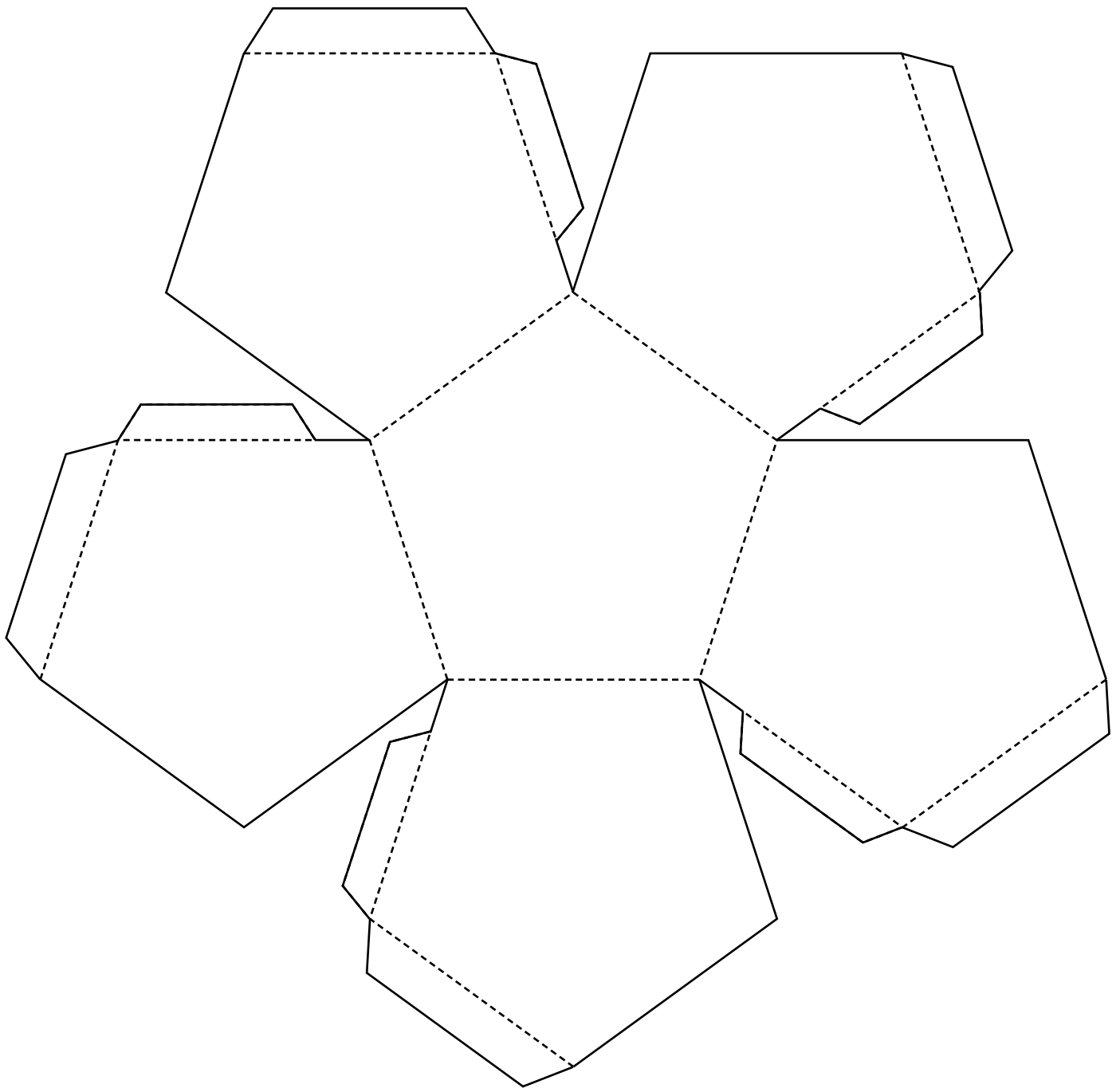
Pyramid volumes - Complex Problems - NP Problems

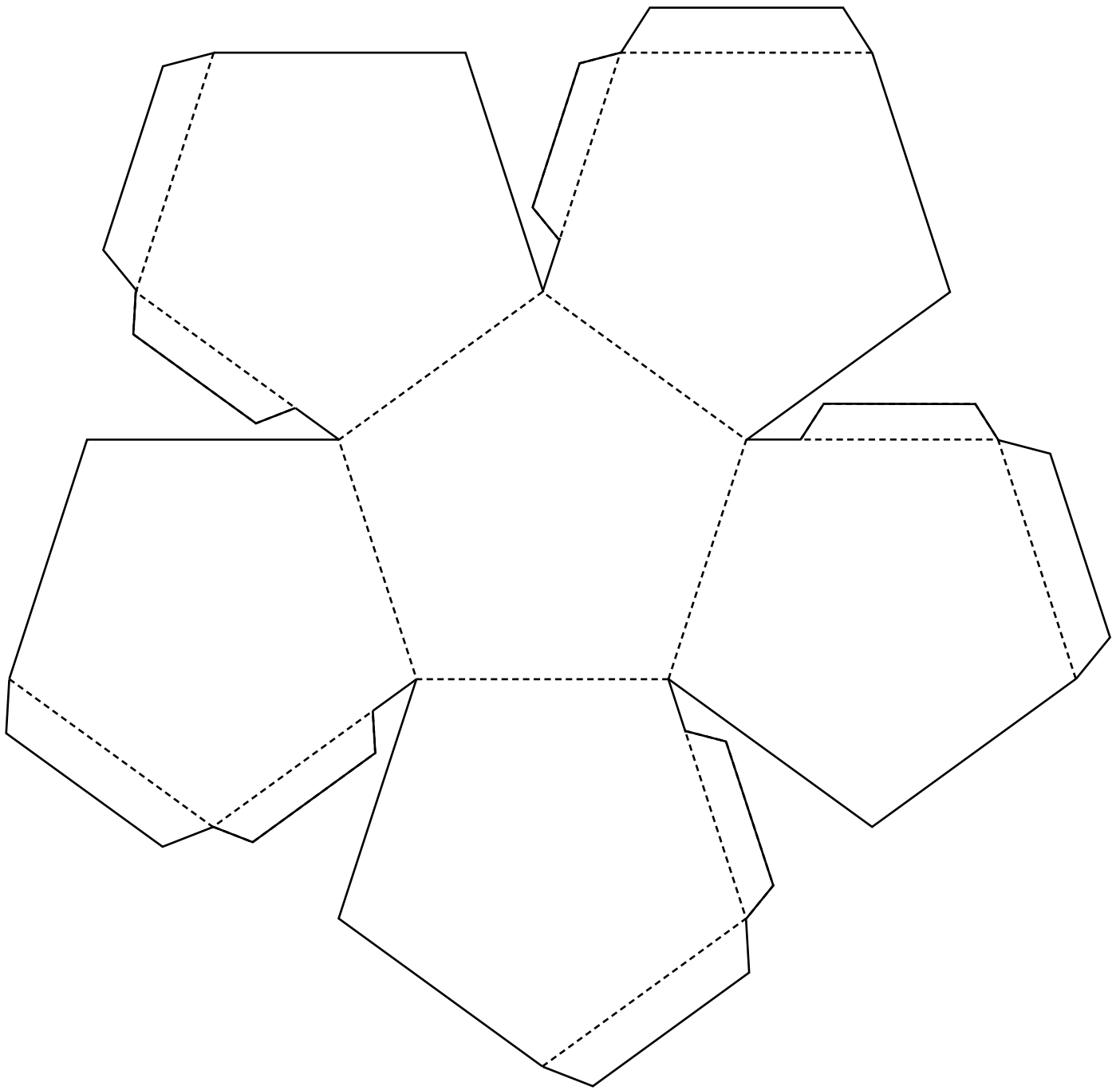


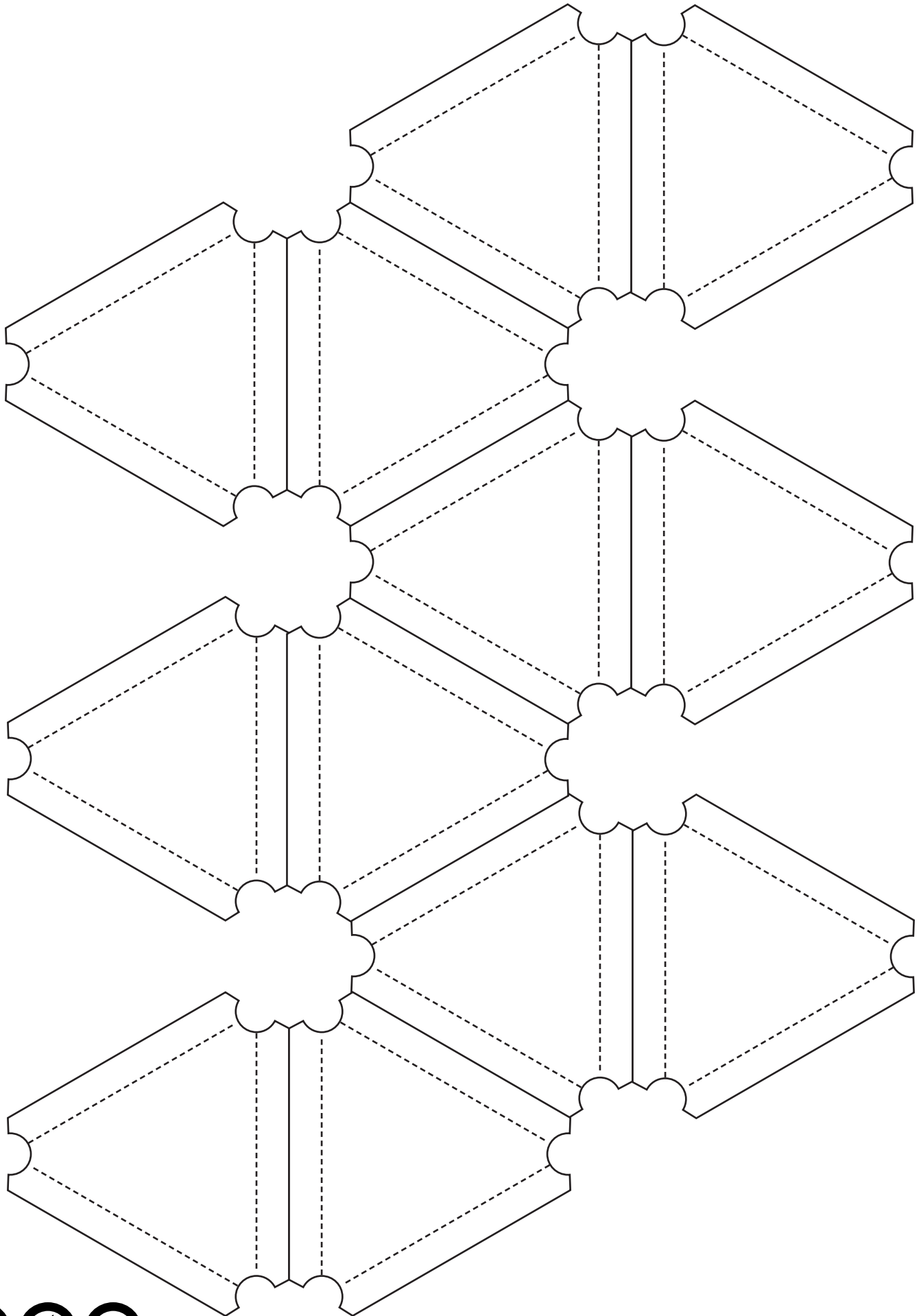


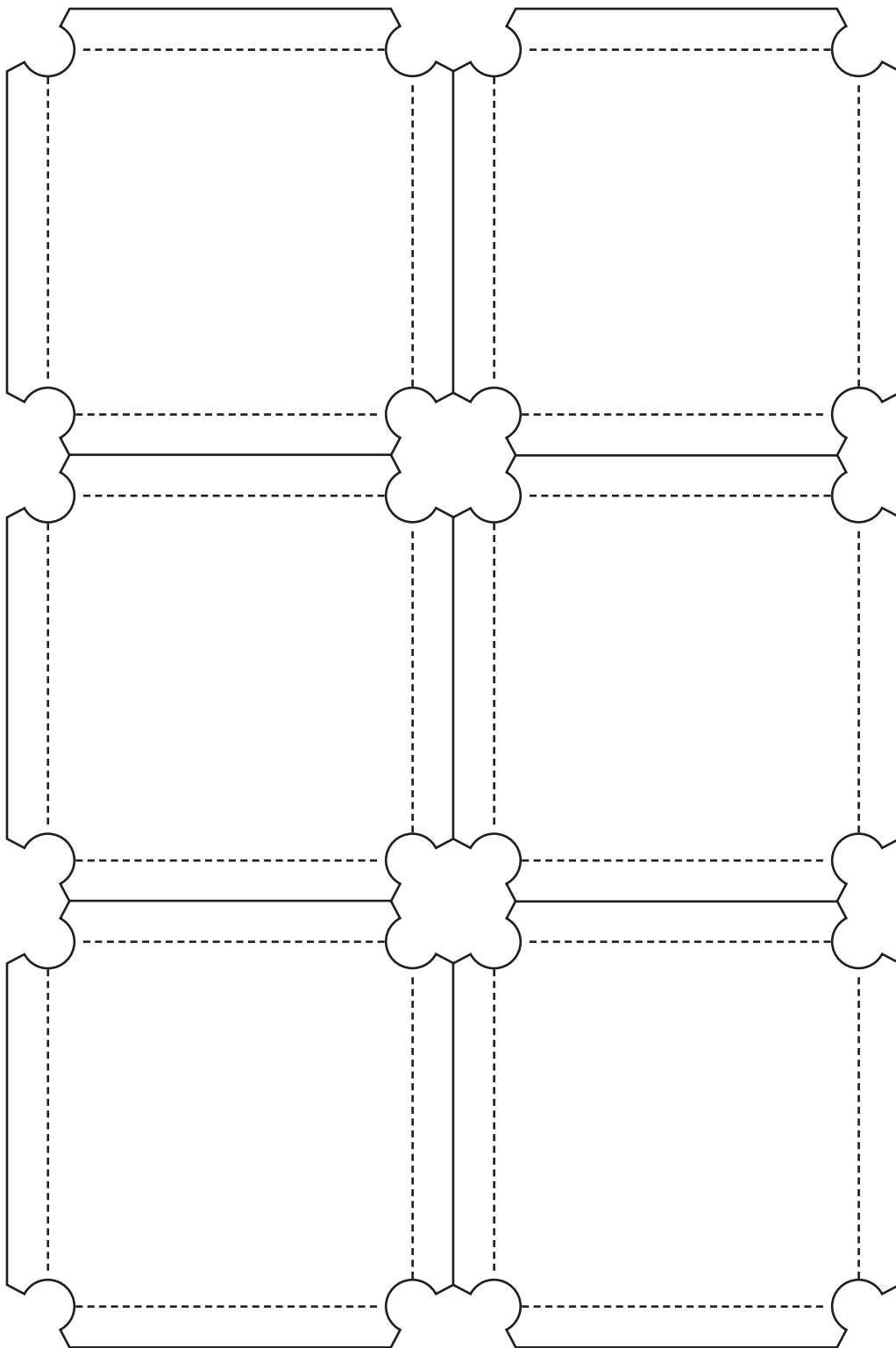


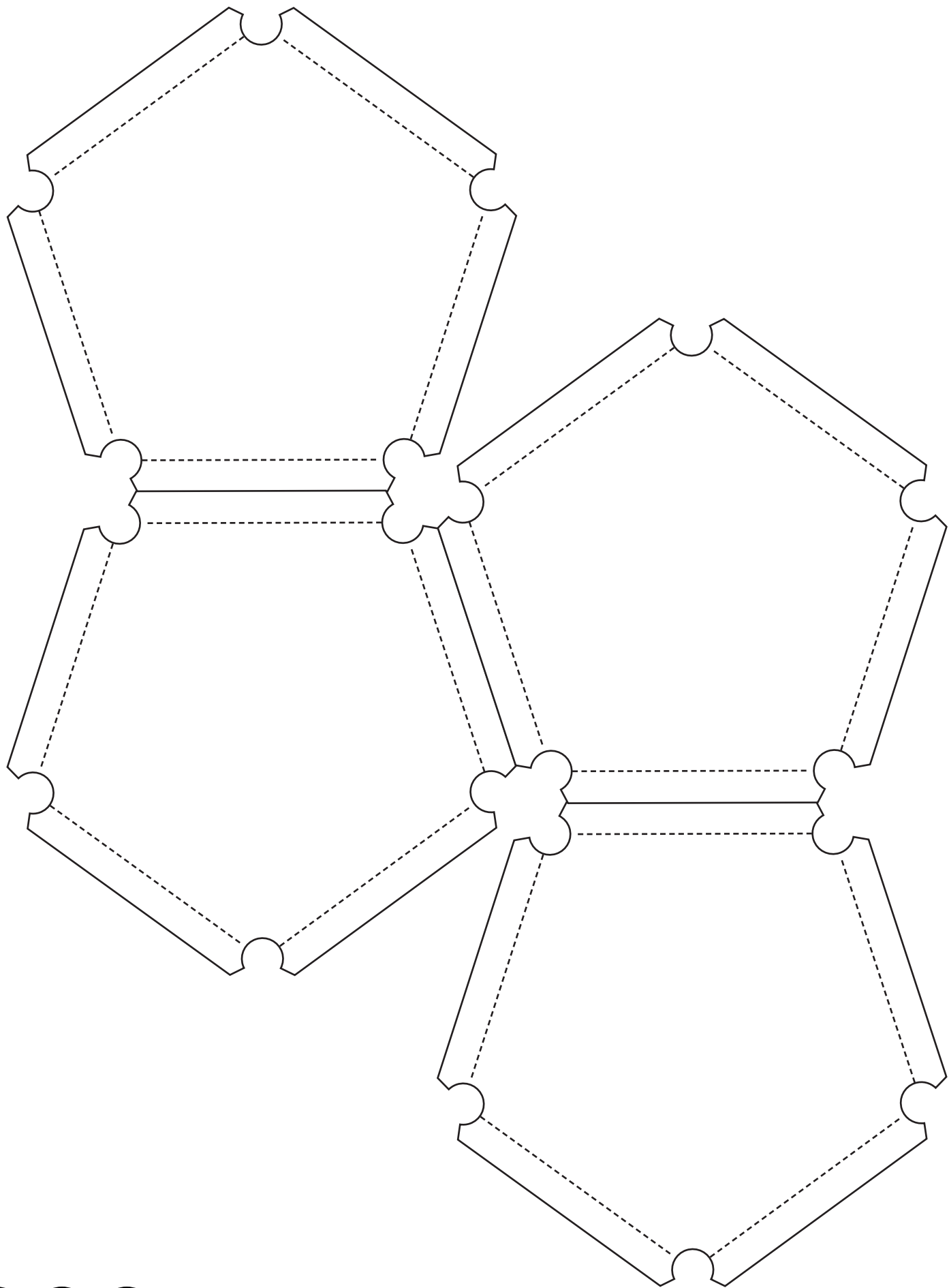


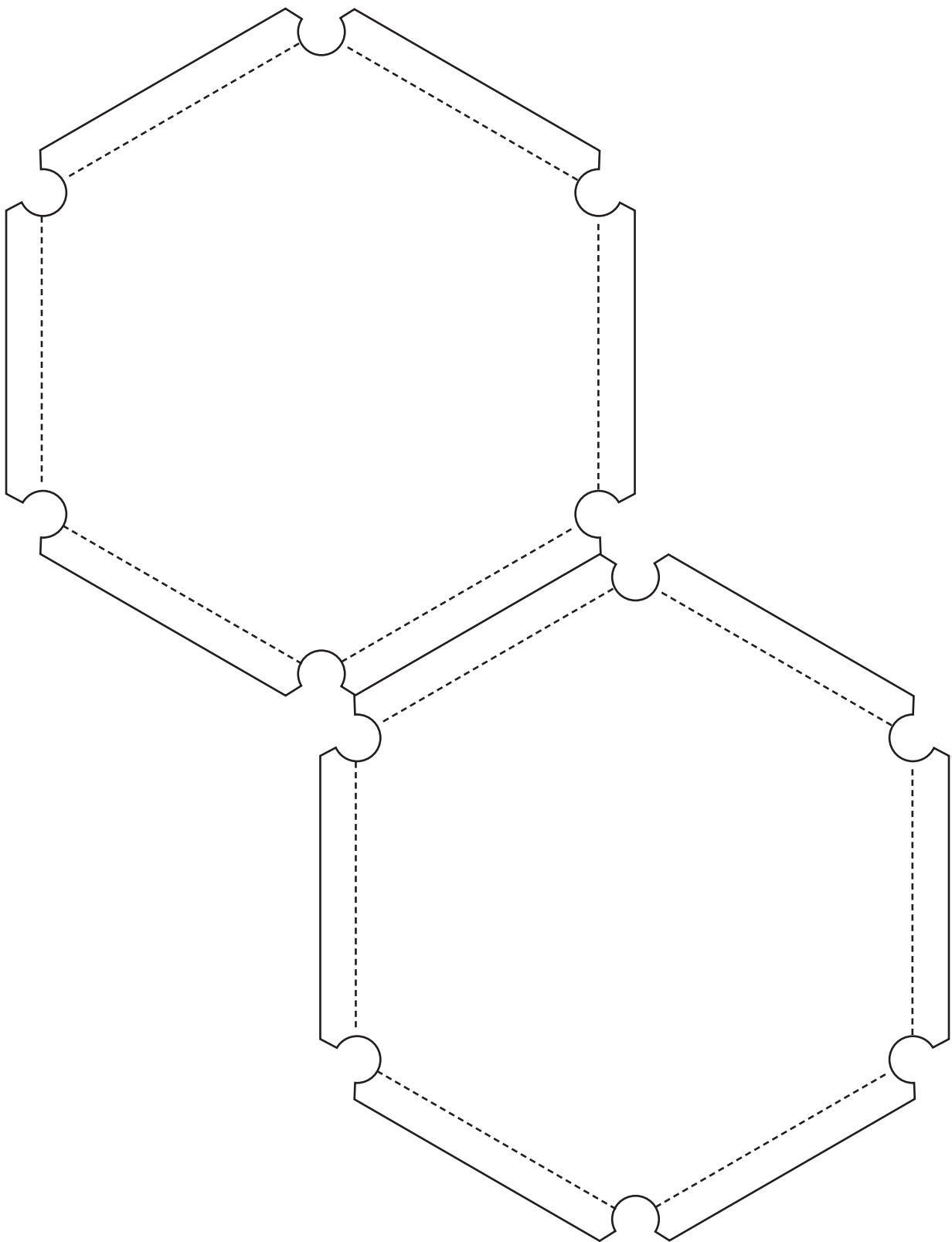
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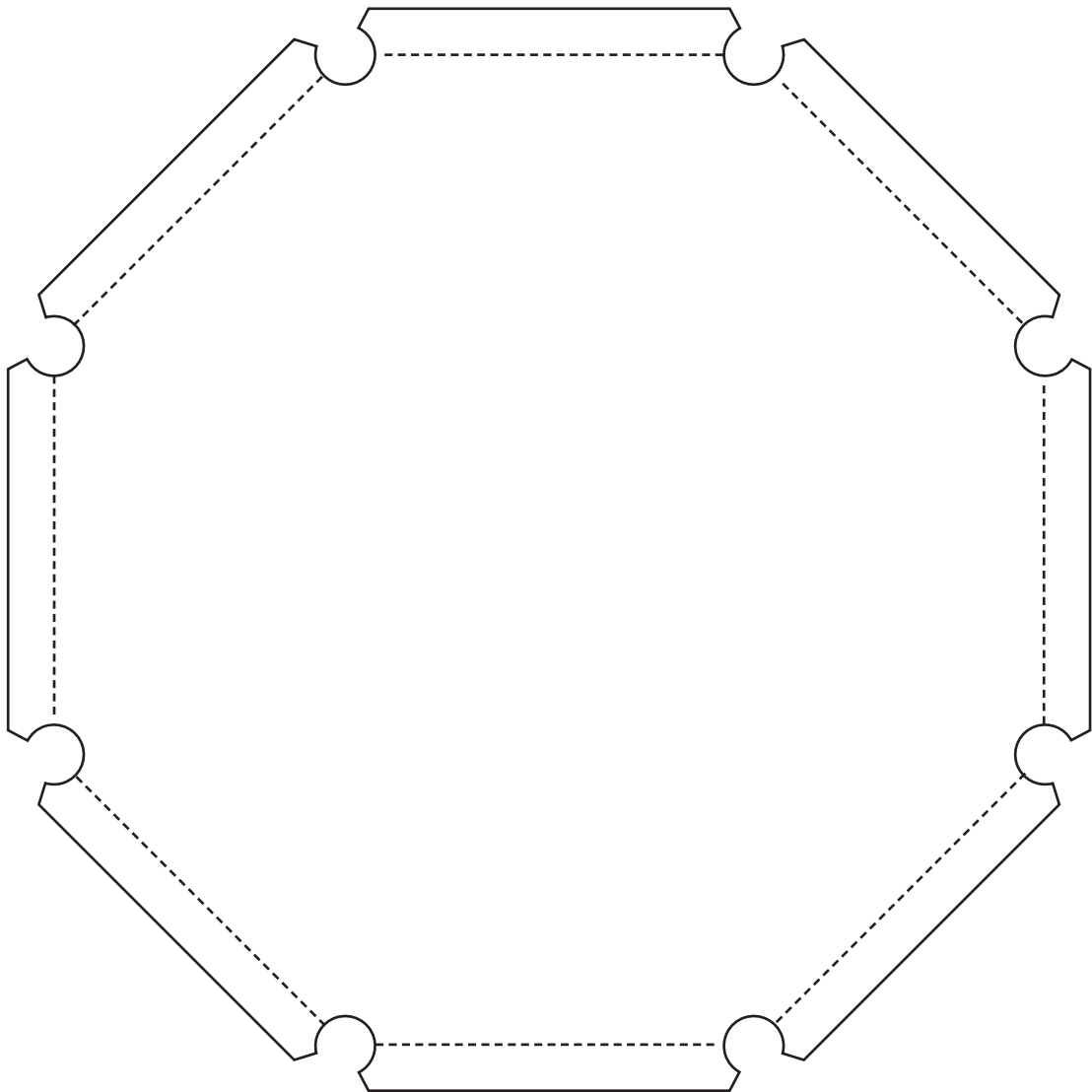
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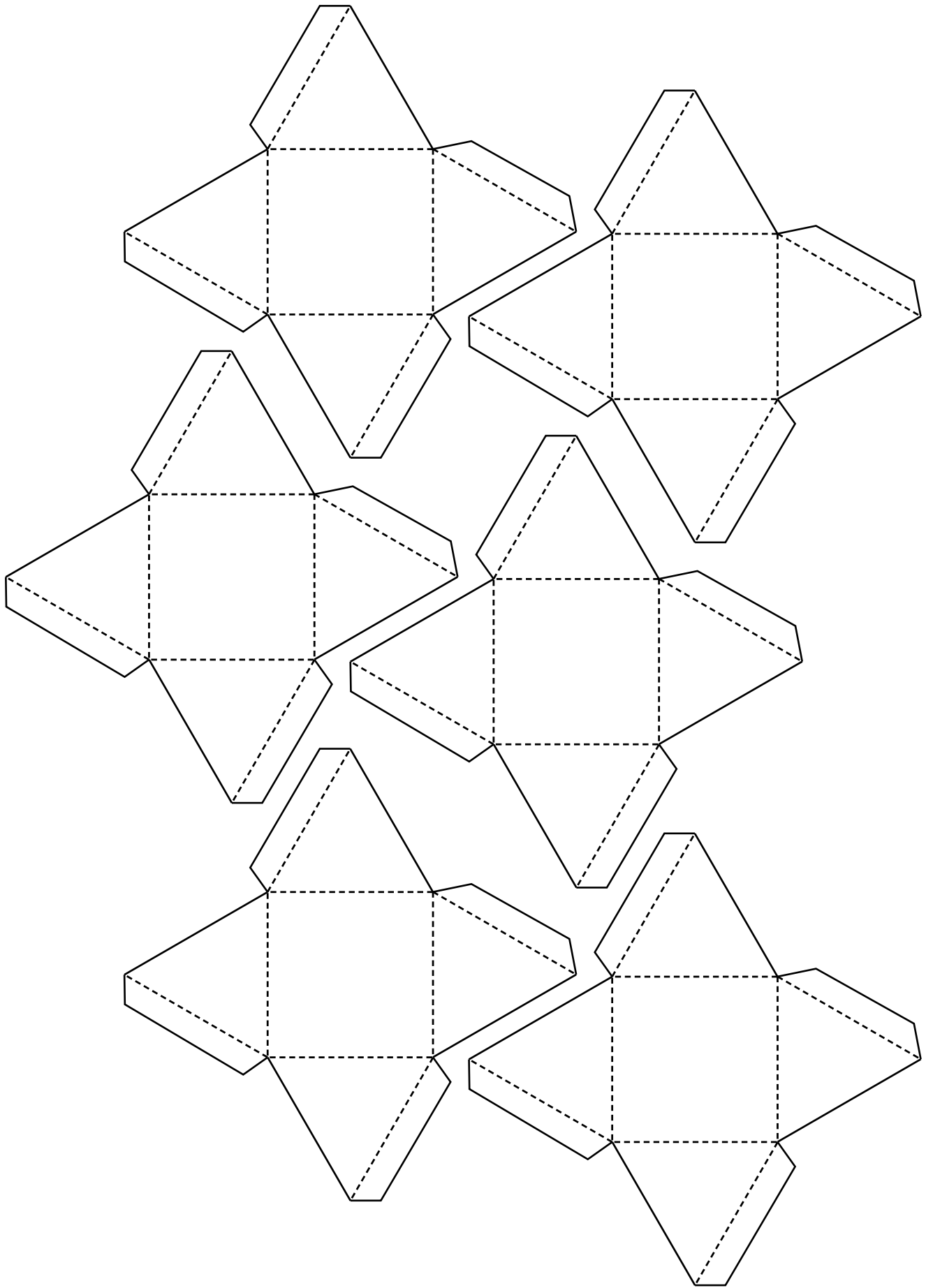


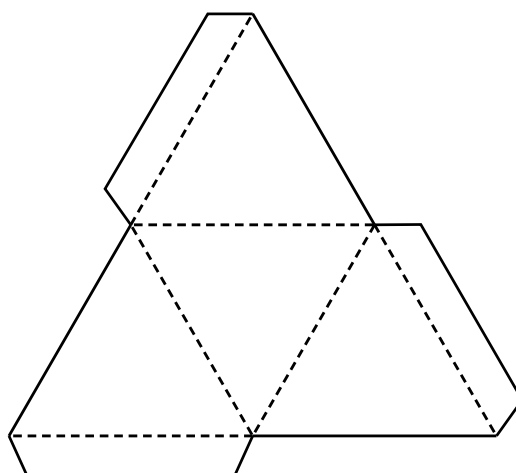
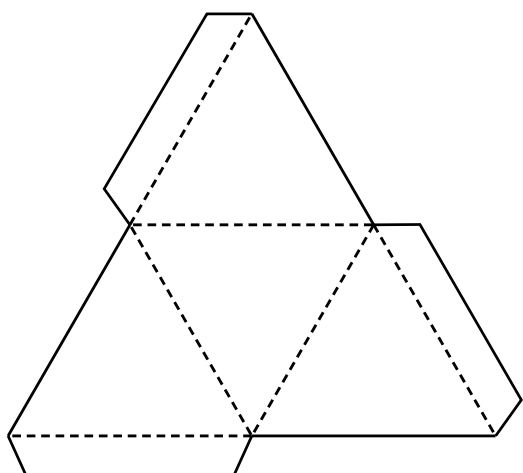
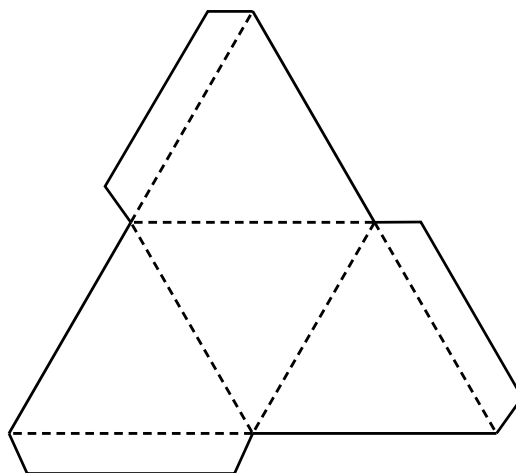
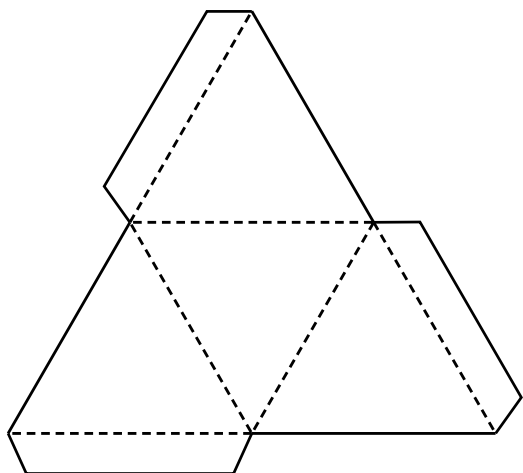
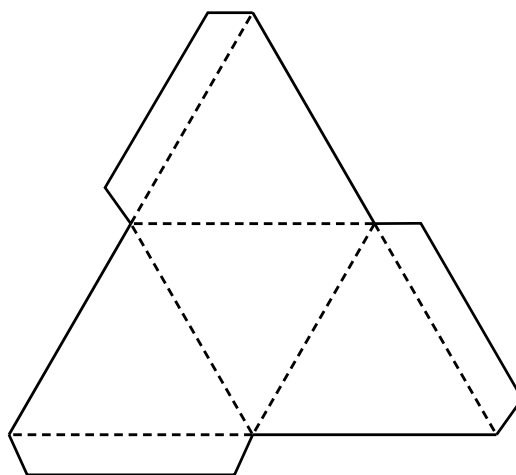
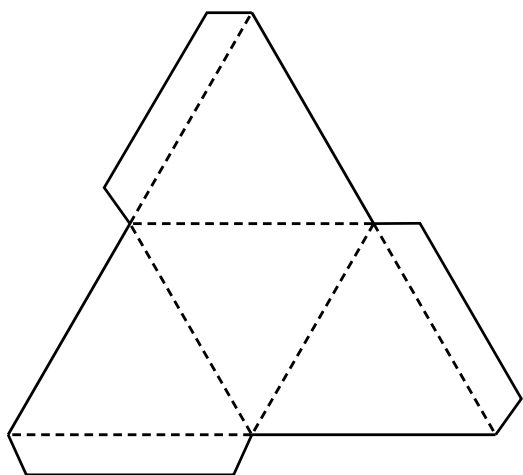














4. Connections



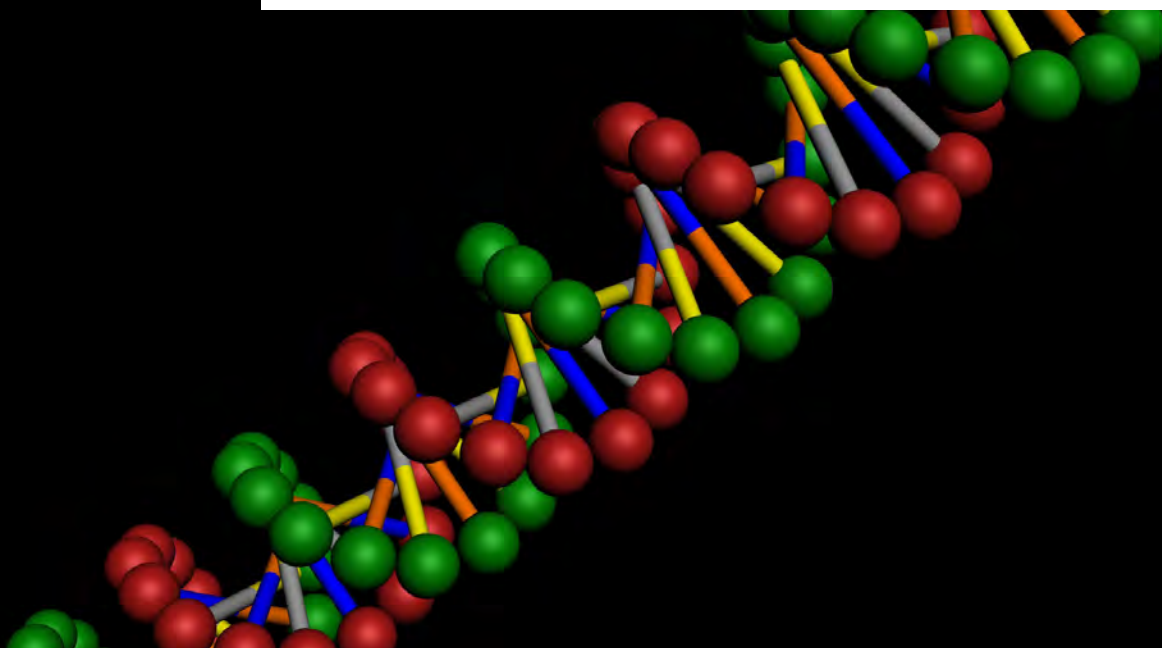
Of a single line



Are 4 colours enough?



Hello! Is that you?



Of a single line



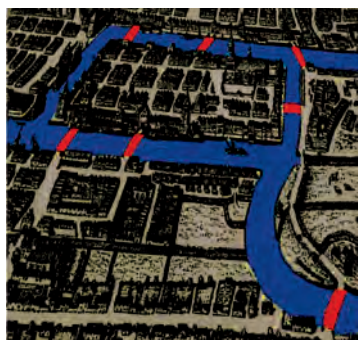
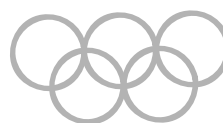
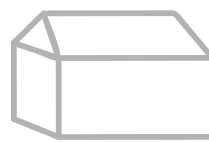
Do it yourself

MATERIAL: Drawing models, 1 pencil

57

Just one line!

Go over each line only once to retrace the drawing without lifting your pencil.
When can you do it? When can't you?



To notice

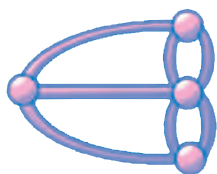
Königsberg*, 1736

Is it possible to take a walk through the town by crossing over every bridge once, and only once?

To solve it, **Euler** summarized the problem to its core information: the city is divided into four districts represented by four points, connected by seven lines which symbolise the seven bridges.

The problem is as follows: on this map, is there a road which passes only once over each line? It is the beginning of **graph theory**.

Euler's answer: how many points are there where an odd number of lines end? There only is a solution if this number is equal to zero or two!



What happens if we add a bridge to link one of the islands to the main land (as it is now the case)?

** Nowadays called Kaliningrad (Russian exclave between Poland and Lithuania)*

Do it yourself

MATERIAL: A set of dominoes, the doubles out

Dominoes - Dominoes

Following the basic dominoes rules, try to make a chain using all the tiles.

Start again leaving aside the dominoes with a 6 on them.

Then, repeat the exercise with those with a 5 on them, etc

Is it always possible? Why not?

Each domino represents the edge of a graph with 7 summits, numbered from 0 to 6.

Each Eulerian path corresponds to a chain of dominoes.

Maths-related jobs

The graph theory is used to model and study important and concrete situations such as telecommunication networks, electronic circuits, distribution networks (water, gas, electricity, post...), numerous logistics problems but also transport and production...

KEYWORDS FOR WEBSITES:

Graphs - Graph theory - Eulerian's path - Euler

Are 4 colours enough?



Do it yourself

MATERIAL: 1 geographical map, 4 colour pencils or felt pens

58

With 4 colours only!!

One or two players.

Try to colour this map using the minimum number of colours as possible.

Game rule:

2 neighbouring countries must have different colours. Do not forget the sea!

The loser is the one who cannot play anymore.



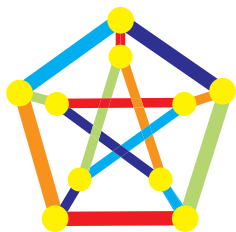
To notice

The 4 colours theorem

The graph theory allowed modelising the problem and therefore reduced the number of cases to study. The analysis of all the possible situations and the proof that only 4 colours are enough was only possible using a computer. It is possible to find an algorithm to colour a map with 6 colours. However, a general solution for the 4 colours problem is yet to be found.

This is a complex problem as the time it takes an algorithm to solve a problem increases exponentially with the number of «countries».

Non-deterministic algorithms, such as genetic algorithms, enable a quicker solution.



Do it yourself

MATERIAL: Polygons (tetrahedron, cube, octahedron, dodecahedron, pyramid...), 4 colour pencils or felt pens

59

60

61

62

63

And what about space?

Build a regular polyhedron (or any other shape) using the 4 colours rule: two neighbouring faces must have different colours.

In the same way, build a map on a polyhedron with a hole in it and try to color it with 7 colours.



Do it yourself

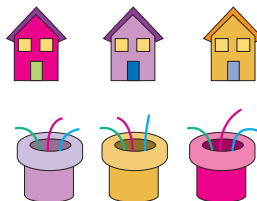
MATERIAL: 1 drawing on a plane and on a torus

3 wells and 3 houses

For each case, try to link each well to the three houses without crossing any of the connections.

Question:

What happens if we add a 4th well and a 4th house?



Question

A wolf, a goat and a cabbage are on the left bank of a river. A man on a boat must ship them across but can only carry one of them at a time. Help him but be careful!

Take Care

The wolf eats the goat and the goat eats the cabbage!!!

Maths-related jobs

The algorithms try to solve general object packing problems following certain rules. They have applications in the setting of tasks such as operating schedules, timetables, exams... but also landline or mobile phone networks, Internet communication networks, secured web data transmissions...

KEYWORDS FOR WEBSITES:

Graphs colouring - Algorithms - Genetic Algorithms

Hello! Is that you?



Do it yourself

MATERIAL: 2 maps, 1 pencil

64

The travelling salesman

A traveller wants to visit 10 towns with as little travelling time as possible. He must start from a town, visit every town once and only once, and come back to the starting point. Help him plan his journey.



Athens

③ Berlin

④ ① Brussels

⑤ ④ ③ Lisbon

④ ① ① ③ London

④ ③ ② ① ② Madrid

④ ① ② ⑤ ② ④ Oslo

③ ② ① ② ① ② ② Paris

① ② ② ④ ② ③ ③ Rome

③ ② ② ⑤ ② ④ ② ③ ② Warsaw

To notice

Distances may be measured in time, travel costs, electricity or water flows...The greater the number of towns, the longer it takes to calculate solutions to this somewhat simple stated problem.

If it takes a computer one microsecond to perform 60 calculation steps for 10 towns, with 100 towns, it would take a computer hundreds of years to perform the 2^{60} steps of calculation (2 multiplied by 2 sixty times). The more complex the algorithm, the more machine running-time it needs.

Do it yourself

MATERIAL: Drawing patterns, Polyhedra to build, 1 pencil, 1 string

62

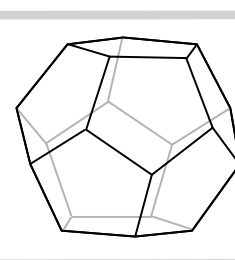
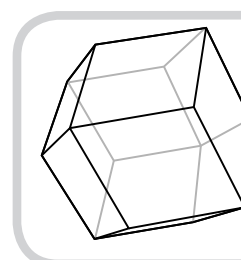
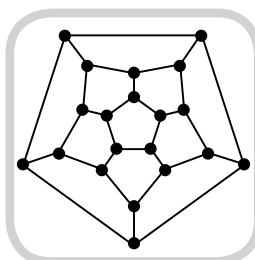
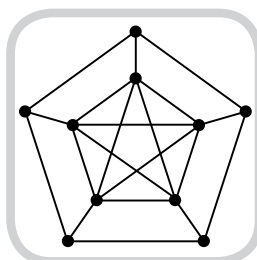
63

65

66

Go around the world

Choose a shape or a polyhedron and try to find a way to visit every vertex once and only once.



Remember

Finding a Hamiltonian path is like finding a path visiting each vertex once. There is still no general solution for this type of problem. **Hamilton** proved that there are solutions for the 20 vertices of a regular dodecahedron (made of 12 pentagons).

Is this the same for the other dodecahedron (made of 12 lozenges)?

Hello! Is that you?

Do it yourself

MATERIAL: 1 chessboard, cut-out pieces, scissors



Checkmate the Queens?

- Try to place eight Queens on the chessboard so that none of them can take another Queen.
- Try to move the King to visit all the squares once (do not use diagonals).
- Try to move the Knight on the chessboard by passing over each square once and only once.

Do it yourself

MATERIAL: Paper & Pencil

From Beef to Lids

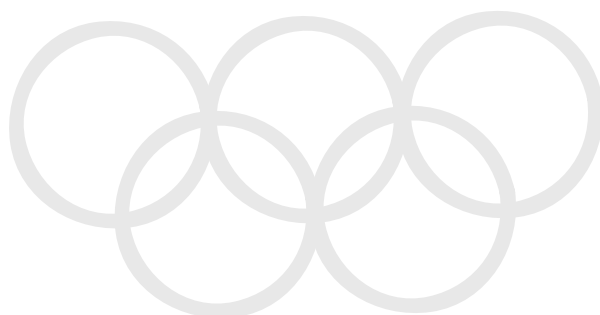
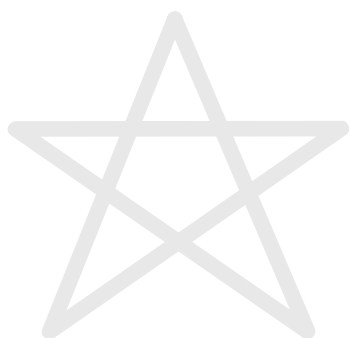
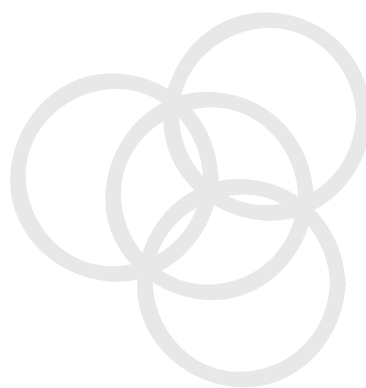
Choose two words with the same number of letters. Try to go from one word to another word changing only one letter at a time, like for example, from BEEF to LIDS or from LADS to TAPS or from ONE to TWO or ...

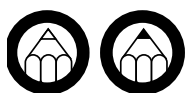
Maths-related jobs

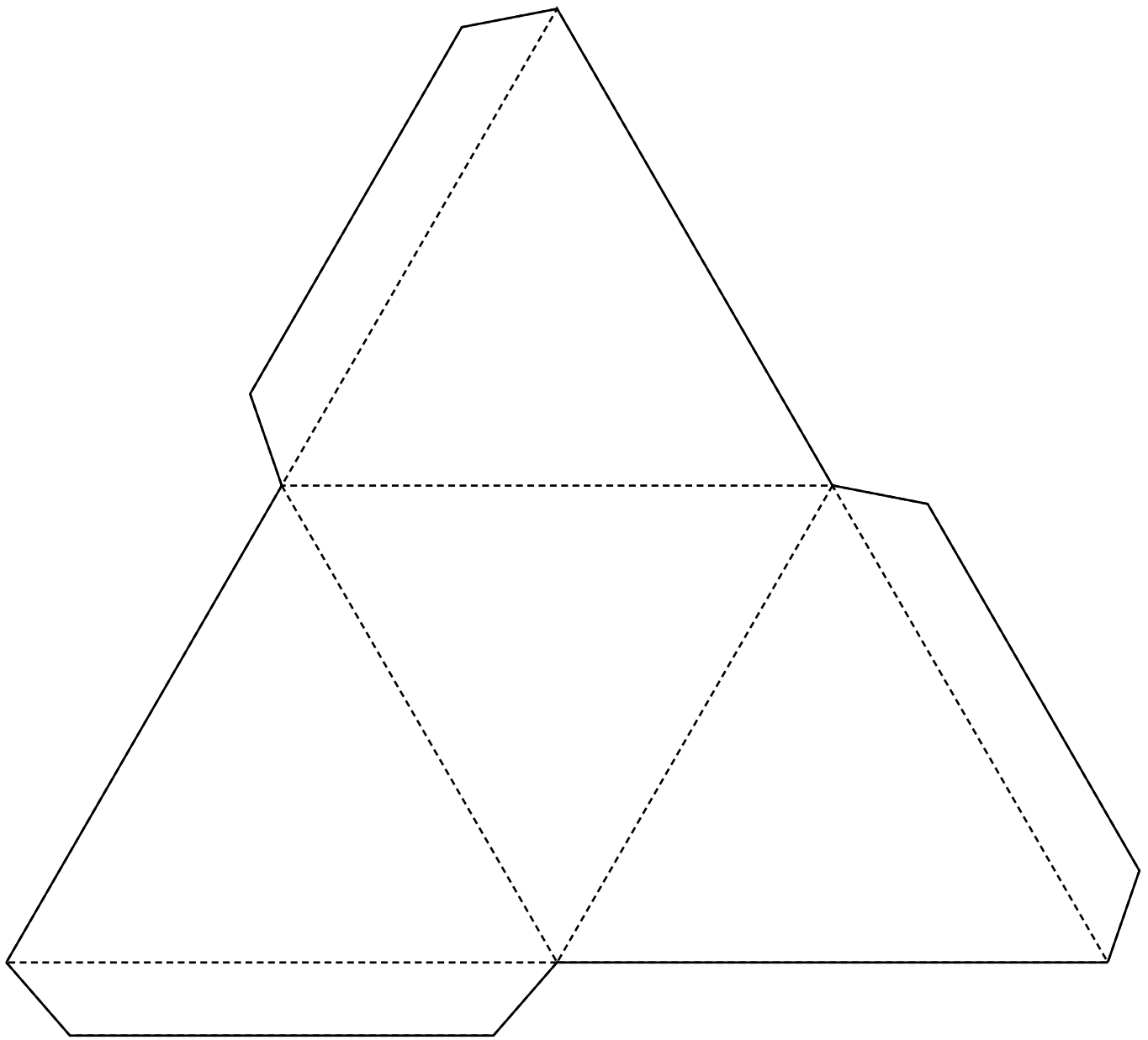
Mathematicians, computer scientists, geneticists have conducted many researches to find efficient algorithms which would solve these complex problems. The sequencing of the 30,000 to 100,000 A-T C-G bases of a DNA molecule is one of these studies.

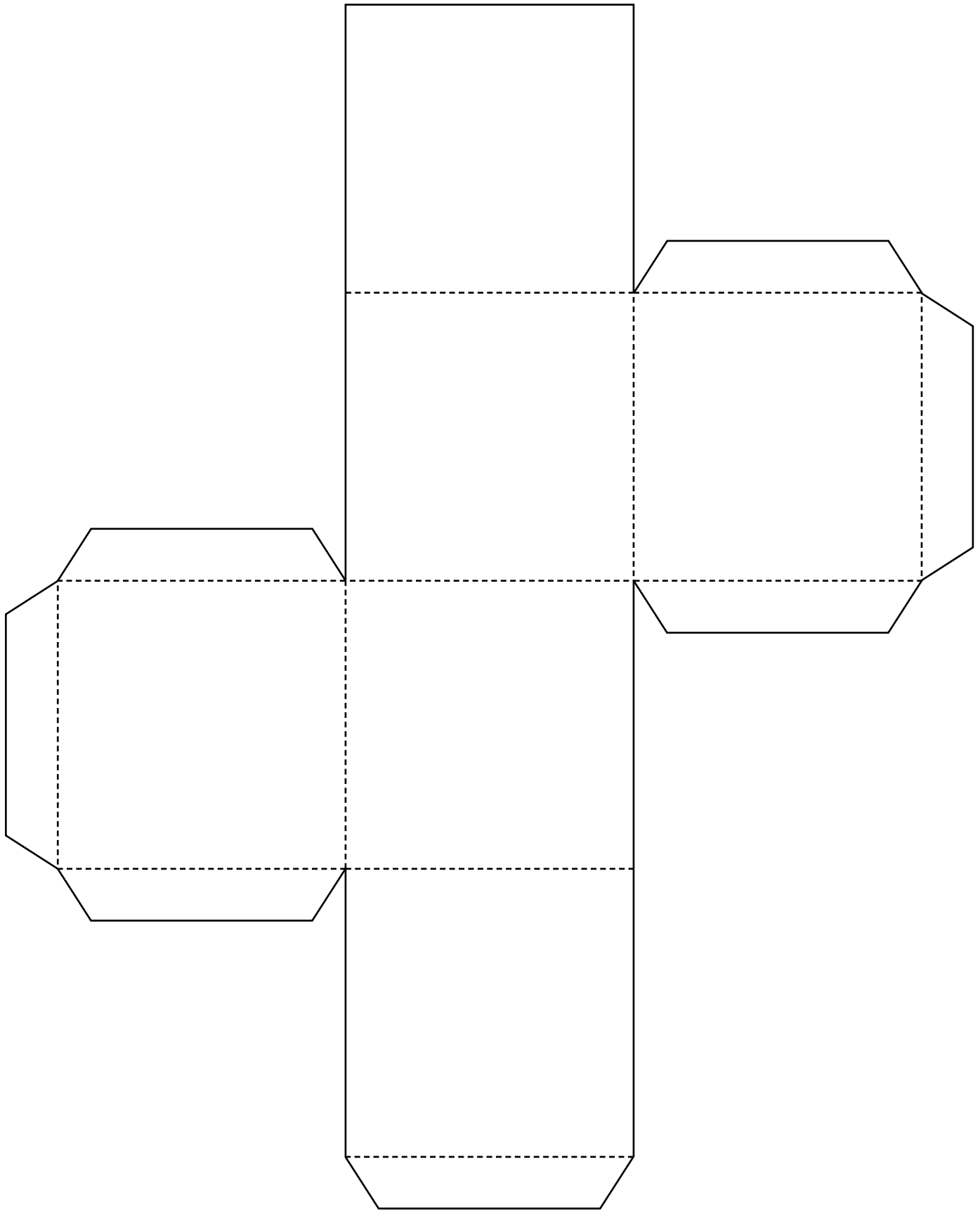
KEYWORDS FOR WEBSITES:

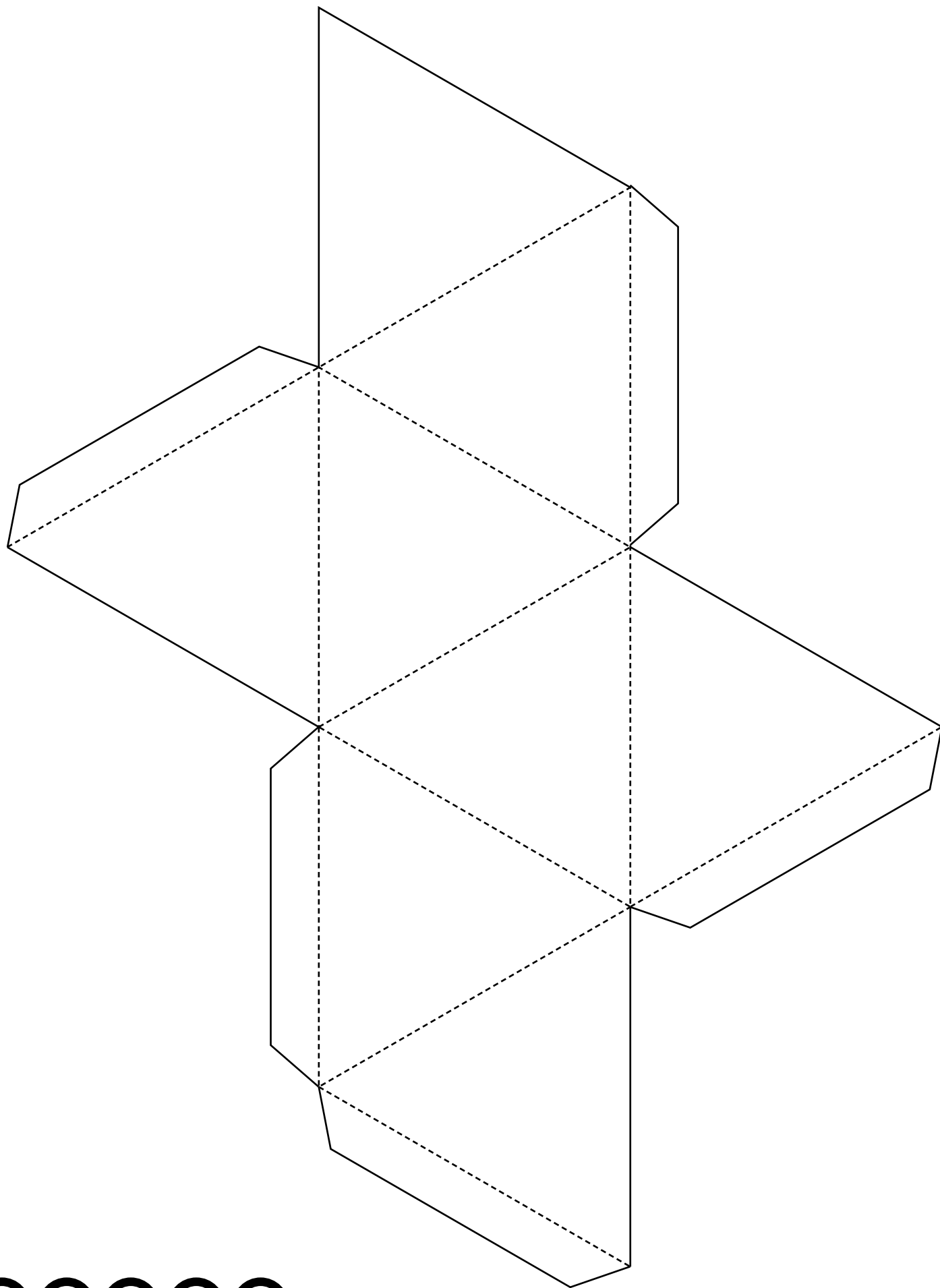
Graphs - Hamilton paths - Travelling salesman - Optimisation

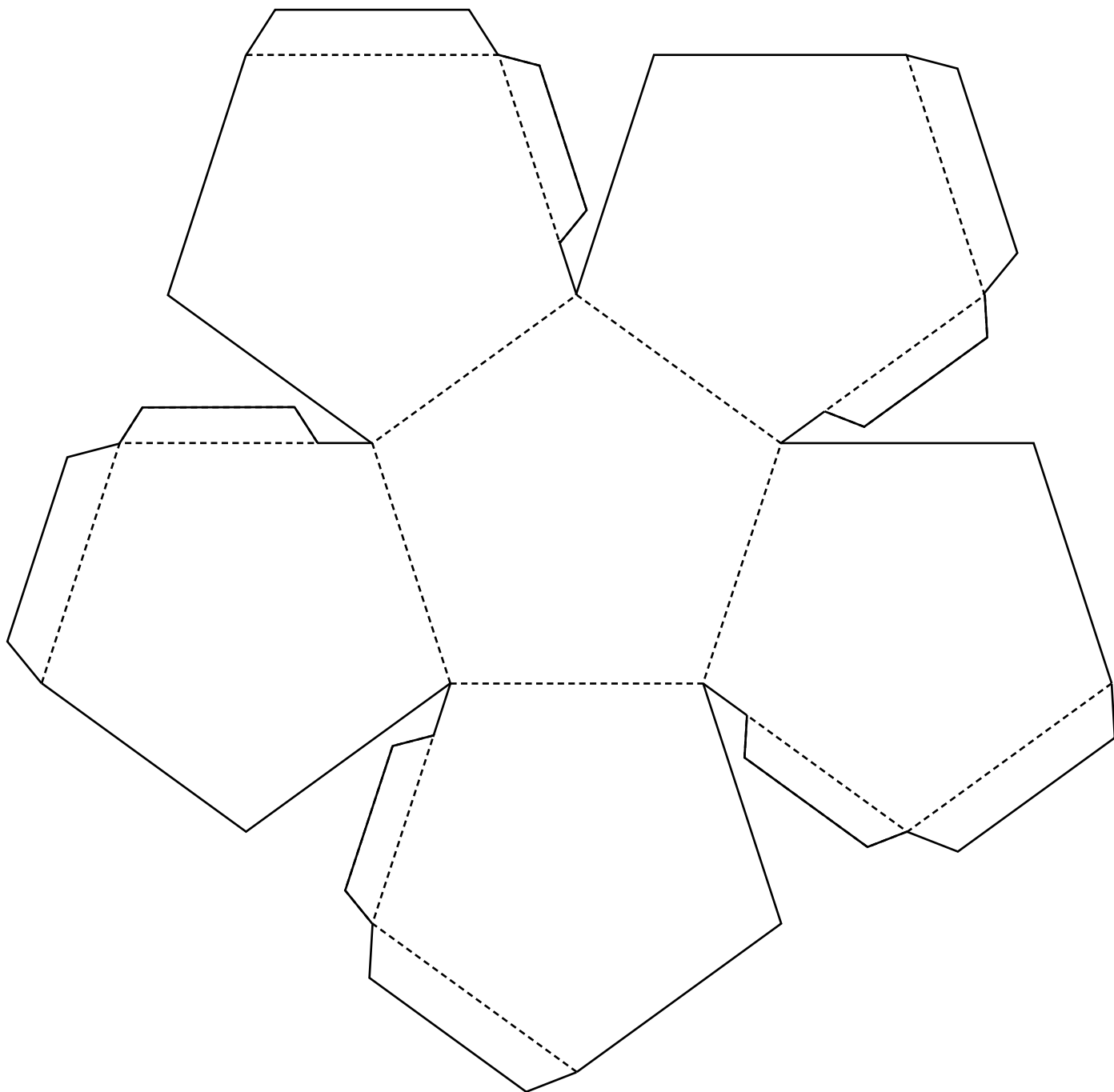






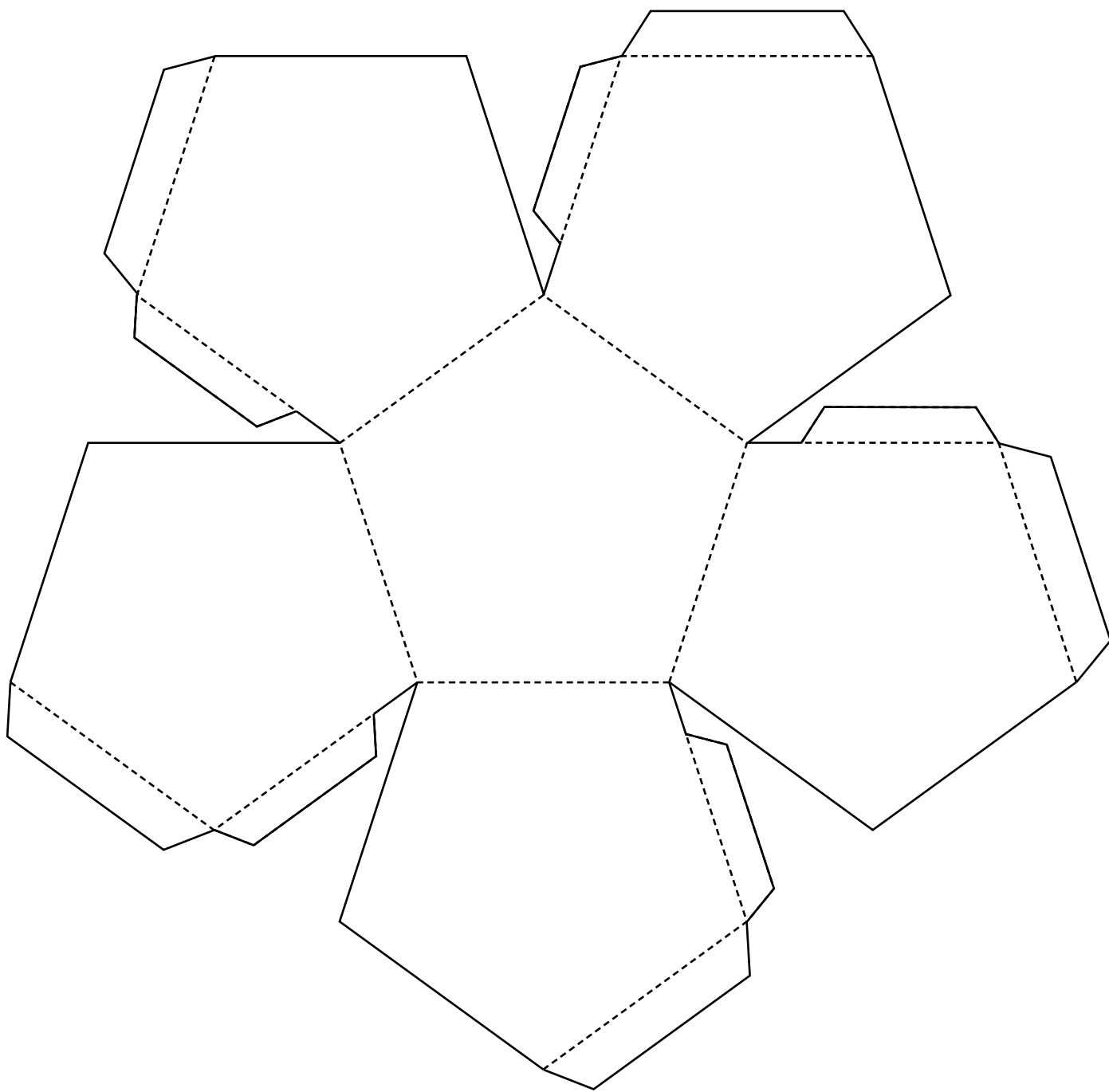






1/2



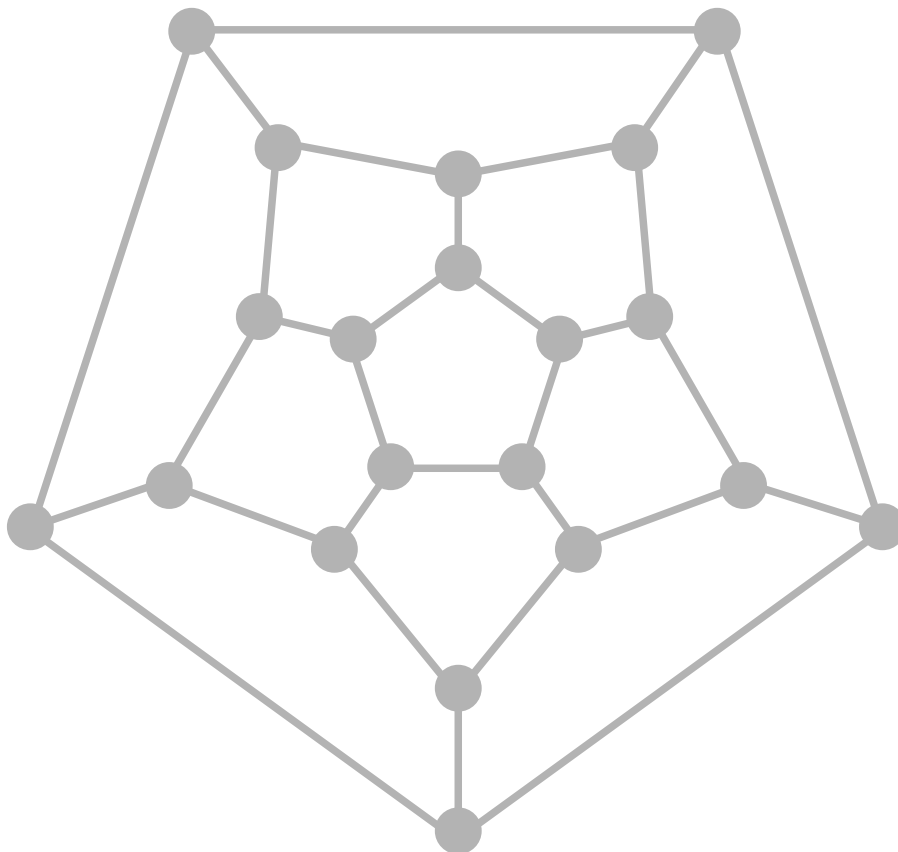
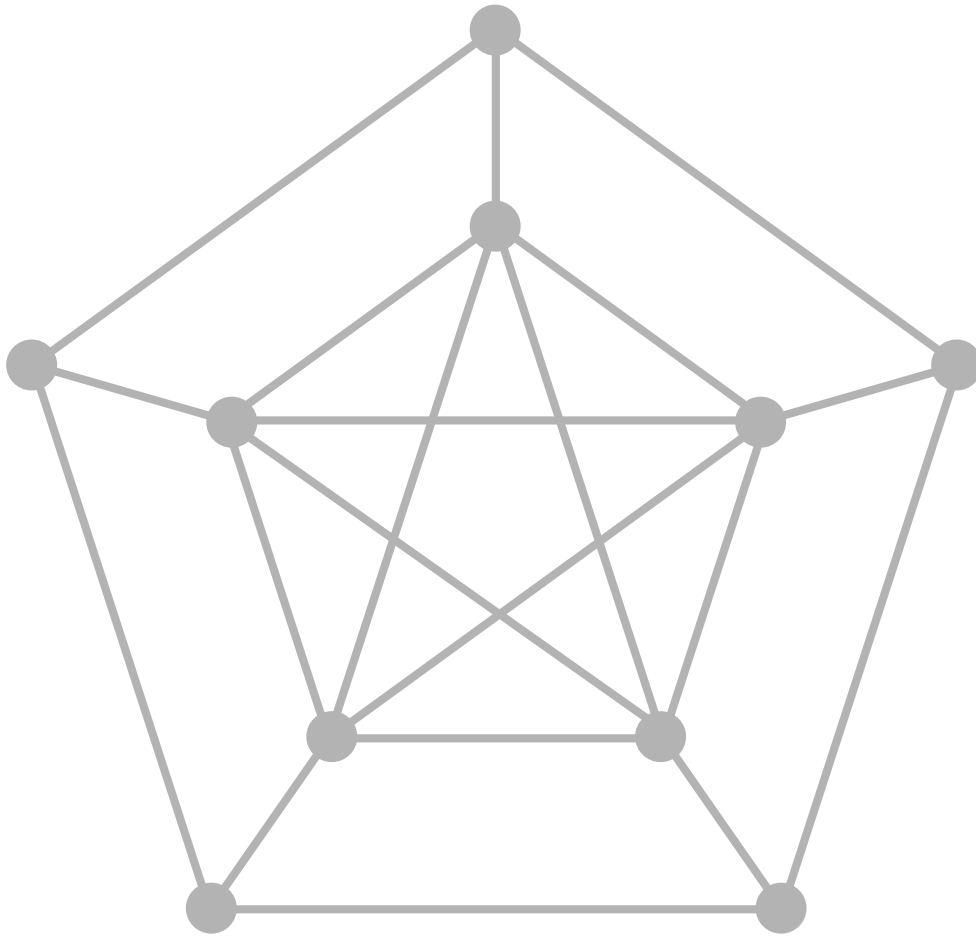
**2/2**

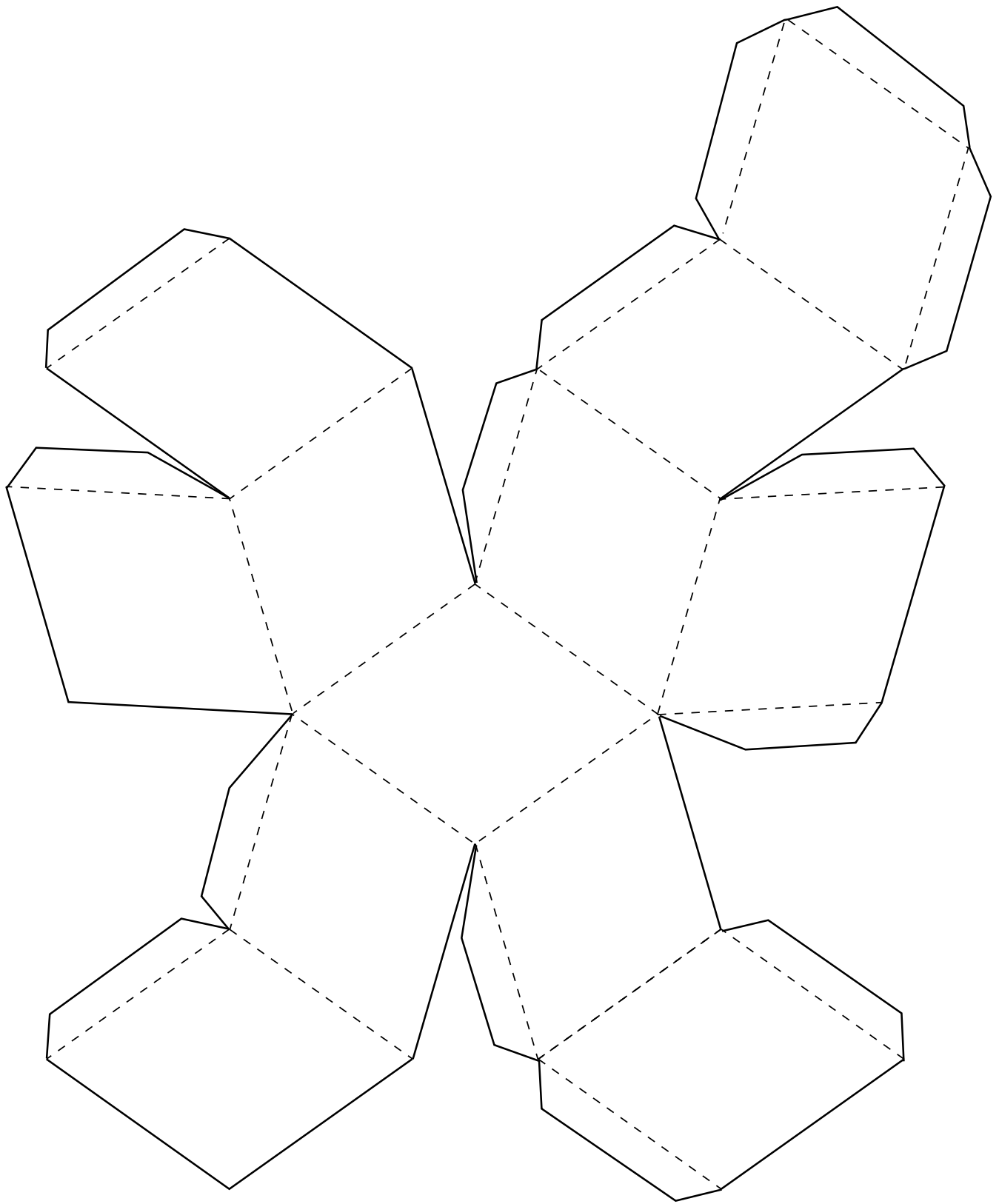


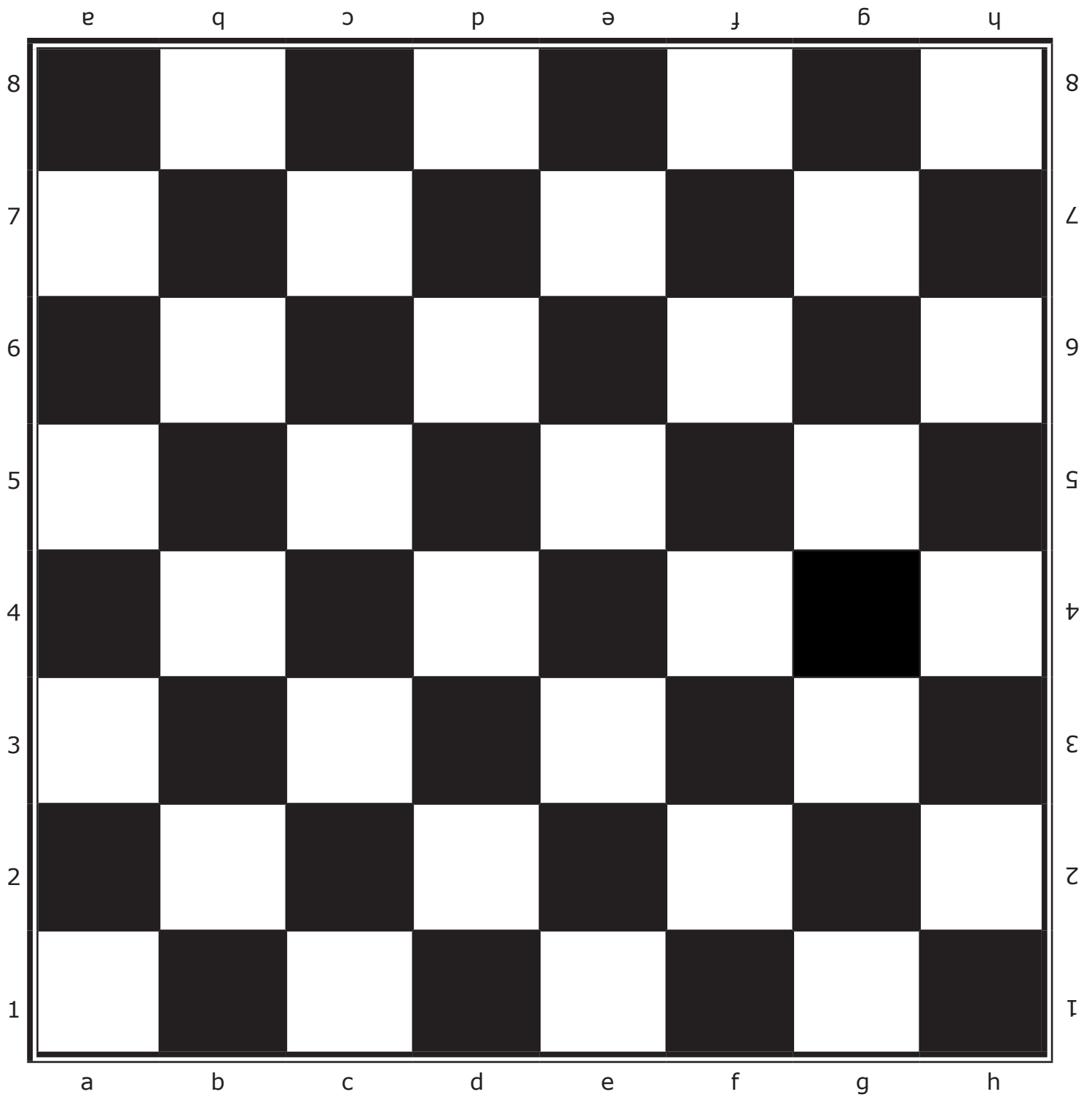
Athens

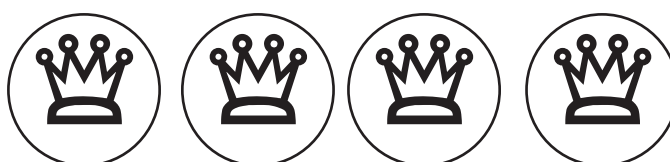
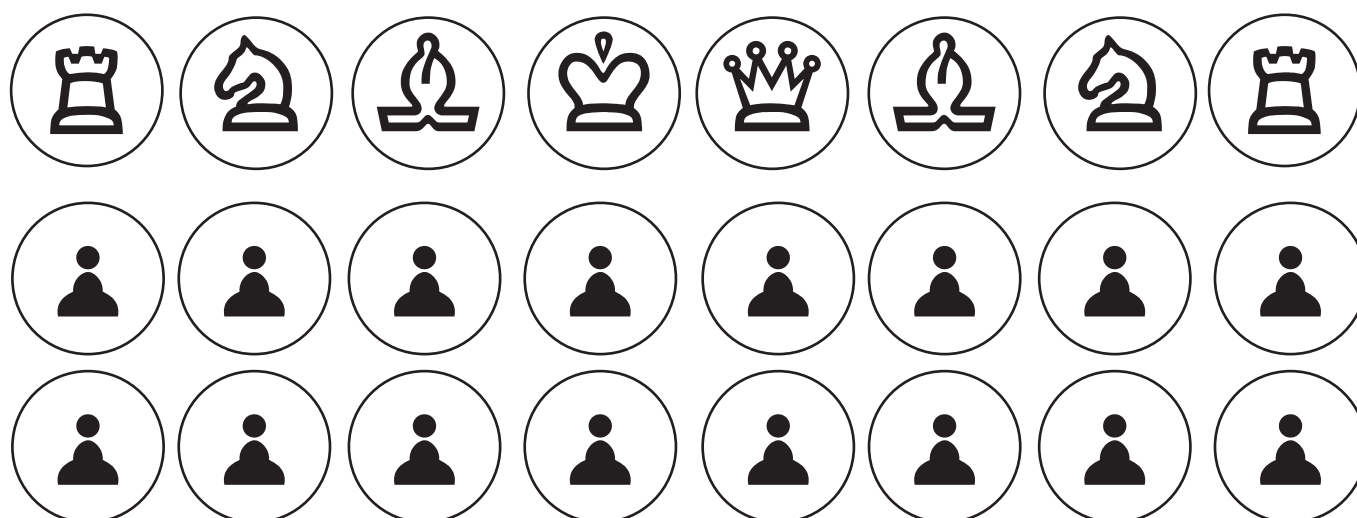
3	Berlin							
4	1	Brussels						
5	4	3	Lisbon					
4	1	1	3	London				
4	3	2	1	2	Madrid			
4	1	2	5	2	4	Oslo		
3	2	1	2	1	2	2	Paris	
1	2	2	4	2	3	3	2	Rome
3	2	2	5	2	4	2	3	2 Warsaw











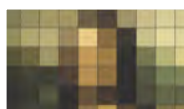
5. Calculating



With your head & hands



Prime numbers



Digital pictures

With your head & hands



Do it yourself

MATERIAL: Your two hands!

Count on your fingers!

9X1... 9X2... 9X3... **9 X 4 =** 

With both hands open facing you, starting from your left hand count to 4 with your fingers and fold down the 4th finger.

- Read **3** fingers up on the left for the tens, and **6** fingers up for the units: **36**

Two numbers between 5 and 10 **6 X 8 =** 

Count to 6 with your left hand. 1 finger held up.

Count to 8 with your right hand. 3 fingers held up.

- Result: **3+1** make **4** tens and **4x2** for the folded down fingers make **8** units: **48**

Two numbers between 10 and 15 **13 X 14 =** 

Count to 13 with your left hand. 3 fingers still held up.

Count to 14 with your right hand. 4 fingers held up.

- Result: **3+4** make **7** tens and **3x4** for the units: **100+70+12=182**

Two numbers between 15 and 20 **17 X 19 =** 

Count to 17 with your left hand. 2 fingers held up.

Count to 19 with your right hand. 4 fingers held up

Then: **2+4** make **6 fifteen** and **2x4** for the units make **90+8=98**

- Result: **15x15 + 98 = 225 + 98 = 323**

To notice

Learning how to count starts with the learning of addition and multiplication up to the 10 times table. Actually, all you need is to learn up to the 5 times table and then, know how to count on your fingers!

As for $(5+a) \times (5+b)$:

For the tens, when you add up the number of fingers held up,
count $10 \times (a+b)$.

For the units, when you multiply the folded down fingers,
calculate $(5-a) \times (5-b) = 25 - 5(a+b) + ab$.

Check that you do have $(5+a) \times (5+b)$.

Check that it also works for the other multiplications.

To use these techniques, you only need to know the squares of 10, 15... Try it for numbers between 20 and 25...

With your head & hands



Do it yourself

MATERIAL: 1 pen, 1 paper or 1 black slate

Mental arithmetic, Quick calculus

Additions, subtractions

Make the students calculate - in their heads - 2-digit numbers additions and subtractions, then 3-digit numbers..., by either writing both numbers, writing one and saying the other one or even by saying both. Make the students describe and analyze the various calculus techniques they used.

38+5
28+18
128+58
289+135...

27-18
66-19
151-28
197-19...

Who are we ?

- Added up we equal 25 but our remainder equals 1.
- We are three consecutive numbers and our sum equals 48.
- I am a 2-digit number. The sum and the product of my digits equal respectively 12 and 14.

Kaprekar Routine (Indian mathematician - 1949)

Take a 4-digit integer, 5294 for instance, and proceed as follows:

K(5294) = 9542 - 2459 = 7083
K(7083) = 8730 - 378 = 8352
K(8352) = 8532 - 2358 = 6174
And K(6174) = !!!

Repeat these calculations with other 4-digit numbers and make hypotheses about the various possible cases.

Multiplications, divisions

- First, make sure the students know the squares of 11, 12, 13, 15, 20, and 25 by heart.
- Now, it's time for the students to practice dividing and multiplying by 5, by 9, by 12, 13, 15, 19, 25, 50 and 100.
- Calculate 46×96 and 64×69 . What do you think of the result? Strange, isn't it? Find more examples.
- Calculate 23×9 and 78×9 . 23 and 78 are said to be associated. Find more examples!

Astonishing multiplications

Calculate, continue and find of others!

1 x 8 + 1 = ...	9 x 9 + 2 = ...
12 x 8 + 2 = ...	98 x 9 + 6 = ...
= ...?	= ...?
1 x 9 + 2 = ...	1 x 1 = ...
12 x 9 + 3 = ...	11 x 11 = ...
= ...?	= ...?

The Syracuse conjecture

- Consider an arbitrary integer N and proceed as follows:
 - If N is even, divide it by 2.
 - If N is odd, multiply it by 3 and add 1.
- Continue the process for each result.

For instance: $20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow 2 \rightarrow 1 \dots$

Apply the operation to other numbers and make hypotheses about the various possible behaviours of the sequences. The greatest value reached in the sequence is called altitude, we speak about time of flight for the length of the sequence before it does pass again under the starting number...). Created in 1932 by Collatz and Hasse (German mathematicians - 1932), this algorithm ended up in a conjecture called the Syracuse problem, or the Collatz conjecture. Is still has not been demonstrated.

With your head & hands



Do it yourself

MATERIAL: 1 pen, 1 paper

Calculus and algorithms

Divide to multiply **57 x 86 = ?**

Russian multiplication

57	86
28	172
14	344
7	688
3	1376
1	2752
	4902

Per gelosia multiplication

x	8	6	
7	6	2	2
	5	4	
5	0	0	0
	4	3	
=	4	9	

Or, near to the traditional technique:

$$\begin{array}{r}
 57 \\
 \times 86 \\
 \hline
 3042 \\
 + 4056 \\
 \hline
 = 4902
 \end{array}$$

To know more

With computers, to avoid writing lines to add and problems of retains... (to be retained!) other techniques of quick calculus are used. It is the field of algorithmic. Thus it is algorithm multiplication of Russian **Anatolii Karatsuba** (1962) :

Then, to calculate: **1234 x 5678**

We cut each 4-digit number out of packages of 2 digits to obtain:

$$\begin{aligned}
 1234 \times 5678 &= (12 \times 10^2 + 34) \times (56 \times 10^2 + 78) \\
 &= 12 \times 56 \times 10^4 + [(12 + 34) \times (56 + 78) - 12 \times 56 - 34 \times 78] \times 10^2 + 34 \times 78 \\
 &= 672 \times 10^4 + [46 \times 134 - 672 - 2652] \times 10^2 + 2652 \\
 &= 672 \times 10^4 + [6164 - 672 - 2652] \times 10^2 + 2652 \\
 &= 6720000 + 284000 + 2652 \\
 &= 7006652
 \end{aligned}$$

It is enough to calculate 3 multiplications of numbers 2 times smaller and some additions moreover but very simple. This algorithm is based on the following algebraic relations:

$$\begin{aligned}
 (ax + b)^2 &= a^2x^2 + [(a+b)^2 - a^2 - b^2]x + b^2 \\
 (ax + b)(cx + d) &= acx^2 + [(a+b)(c+d) - ac - bd]x + bd \\
 \text{and } 4ab &= (a+b)^2 - (a-b)^2
 \end{aligned}$$



Do it yourself

MATERIAL: 1 pen, 1 paper

Quick calculus and proportionality

How to calculate quickly:

- In 6 h an air-conditioning consumes 7 kw. How much does it consumes in 18 h?
 In 9 h an air-conditioning consumes 18 kw. How much does it consumes in 108 h?
 In 32 h an air-conditioning consumes 27 kw. How much does it consumes in 8 h?
 In 21 h an air-conditioning consumes 17 kw. How much does it consumes in 90 h?


Double proportionality

Supplement tables like:

Potato consumption in a school (in kg)

		NUMBER DAYS			
		1	20	40	100
NUMBER PUPILS	1	0,1	2		
	5	0,5		20	
	10	1		40	
			30		150
	20	2		80	

Area calculus of a rectangle (in cm^2)



		HEIGHT					
		1	4	5	6	10	12
LENGTH	1	1					12
	2	2		10		20	
	3						
			16				
	5	5			30		

Area calculus of a triangle (in cm^2)

		HEIGHT				
		1	3	5	8	15
BASE	1	0,5	1,5	2,5	4	
	2,5		3,75			
	3					
	5		7,5	12,5		37,5

Do it yourself

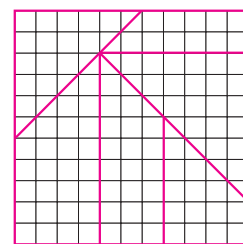
MATERIAL: Pieces of puzzle, 2 par groupes d'élèves, 1 squared paper, ruler, pen, scissors

78 79

Calculus & geometry

Puzzles to increase

With these 6 parts reconstitute a square. Then build the same puzzle in larger by complying with the following rule: the trapezoids of which the height measures 4 cm must be increased to have a 7 cm height. When you finish, you will have to be able to reconstitute a great square with the 6 increased parts.



Going further

Thales and the proportionality

Photography holidays:

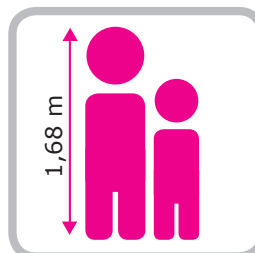
Dad and my two brothers



My little brother and me



My mother and my sister



My father and my sister



Which is my size?

With your head & hands



To notice

The proportionality is an essential axis of mathematics learning and other sciences. It is also a tool very present in the daily life. It makes possible to introduce the multiplication and division at school. It is especially essential to understand the relations between sizes provided with a measuring unit, in physics and other sciences.

The properties used are:

$$f(x + y) = f(x) + f(y)$$

$$f(a \cdot x) = a \cdot f(x)$$

$$f(a \cdot x) + f(b \cdot y) = a \cdot f(x) + b \cdot f(y)$$

Two mathematical functions are in action in these calculus:

The «scalar» function

$$\begin{array}{cc} 6 \text{ (h)} & 7 \text{ (Kw)} \\ 18 \text{ (h)} & ?? \text{ (Kw)} \end{array}$$

The function «with dimensions»

$$\begin{array}{cc} 9 \text{ (h)} & 18 \text{ (Kw)} \\ 108 \text{ (h)} & ?? \text{ (Kw)} \end{array}$$

Do it yourself

MATERIAL: 1 pen - 1 paper

Approximate calculus

Order of magnitude

Make estimate the order of magnitude of calculus checked then with or without calculator. Thus, the result of each one of these calculations it lies between 0 and 1, between 0 and 0,1, between 0 and 0,01, between 1 and 2, between 1 and 10?

$$\begin{array}{l} 125 \div 28 = 0,4464 \text{ or } 4,4643 \text{ or } 44,6428 \text{ or } ??? \\ 28 \div 1275 = 0,0022 \text{ or } 0,0220 \text{ or } 0,2196 \text{ or } ??? \\ 357 \div 176 = 0,203 \text{ or } 2,028 \text{ or } 20,284 \text{ or } ??? \\ 41,84 \times 2,25 = 9,414 \text{ or } 94,14 \text{ or } 941,9 \text{ or } ??? \\ 1/(1+\sqrt{2}) = 0,414 \text{ or } 2,142 \text{ or } 4,142 \text{ or } 21,421 \text{ or } ??? \end{array}$$

Do it yourself

MATERIAL: 1 pen, 1 paper or calculator or computer

My computer tricked me!

Choose a number between 0 and 1. Multiply it by 2.

- If the result is lower than 1, multiply it by 2 again.
- If not, subtract 1 and multiply the result by 2.

And repeat again 60 time.

Remade same calculus with a very nearby number. What do you note?

- Choose a number even more nearby and start again calculus.
- Also try with numbers such as

$\sqrt{2}-1$, $\sqrt{3}-1$ ou $\pi-3$ and with very close decimal numbers.

Example

0,3	0,305
0,6	0,61
0,2	0,22
0,4	0,44
0,8	0,88
0,6	0,76
0,2	0,52
0,4	0,04
0,8	0,08
...	...

For counting, we use intergers and decimals.

At the market, it is better to know to do fast mental or approximate calculations, even if you have a calculator. The computer itself uses only decimal numbers to a few tens of decimal points. The laws of mathematical are no longer respected and often lead any more to errors.

Maths-related jobs

Some techniques of calculation can be used in the daily life. Others are researched by mathematicians and computer specialists to allow computers to calculate always faster and farther or to verify very fast the exactness of bank cards.

KEYWORDS FOR WEBSITES:

Calculus - Quick calculus - Mental calculus - Approximate calculus Order of magnitude - Calculus algorithms



Do it yourself

MATERIAL: 3 grids with integers displayed over 6, 20 & 30 columns, 1 pen, 1 gum

80

81

37	2	3	4	5	6	31
2	17	8	9	10	11	13
3	4	5	6	7	3	29
4	19	8	9	10	11	23
5	6	7	8	9	10	27
6	7	8	23	4	5	6
43	4	5	6	47	4	5

On the trail of the prime numbers!

An integer (other than 1) is called 'prime' when it can only be divided by 1 or by itself.

For each table:

- Cross out 1, then circle the first un-crossed number: '2'.
- Cross out all the multiples of 2 located (after 2).
- Circle the next un-crossed number: it's 3 who is prime.
- Repeat the process until there is no more number to cross out.

According to the table used, observe the properties of this technique. From which prime number did you stop? What can you say?

This technique, called the Sieve of **Eratosthenes** enables to show the prime numbers of the list.

Do it yourself

MATERIAL: 3 dices

Let's play 421

Roll the dice. Write the 3 numbers on the board. The player with the highest 3-digits prime number wins. Each student must be able to say if a given number is prime, or not.



To notice

Divisibility criteria...

An integer is:

- divisible by 2 if the last digit is a multiple of 2,
- divisible by 4 if... by 8 if...
- divisible by 3 if... (also works for 9). Do you know why?
- divisible by 5 if...
- divisible by 11 if...
- and what about by 7? and by 13?

Do it yourself

MATERIAL: A list of prime numbers

The big family of prime numbers

In the family of prime numbers:

- Find twin primes (that is to say two consecutive odd prime numbers) as 3 & 5, 5 & 7, 11 & 13... And what about triplets? Conjecture not demonstrated: there is an infinity of them!
- Find cousins (they differ by 4). There should be as many cousins as twins.
- Then, find sexy ones (differ by 6).
- Check if the product of the first prime numbers + 1 is also a prime number. This proves that there is an infinity of prime numbers, although the bigger they are, the rarer they get.
- Check that any even number (lower than 100 for example) can be written as the sum of two primes. (This is called the Goldbach's conjecture. There is big prize for who proves the conjecture!).

Prime numbers



Do it yourself

MATERIAL: 1 paper, 1 pen

The Big Family of Prime Numbers

- Write 10101, 1001, 101, 11 as products of prime numbers.
Repeat the process for 304, 305, 403, 404, 504, 604, ... Any integer can be the product of prime numbers with one unique construction.
- Beside the big family of prime numbers, there are perfect numbers. These numbers are called perfect as they equal the sum of their divisors, like $6 = 3+2+1$.
Please note that $1/6 + 1/3 + 1/2 = 1$ and that this can be applied to any perfect number.
- Find integers, products of 2 different primes. The greatest number wins!

Secret codes used nowadays are based on these products of prime numbers. The numbers used to create the codes are so great that even the most powerful computers would not have enough calculation time to break them.

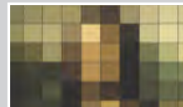
- Find the GCF and the LCM (Greatest Common Factor and the Least Common Multiple) of 28 & 70, of 330 & 900, of 276 & 483,...
- On these examples, check that: $\text{GCF}(a,b) \times \text{LCM}(a,b) = ab$.

Maths-related jobs

Research on prime numbers and their relationships with computer science are many and numerous questions are yet to be solved. Computer and information networks security (cryptography, error-correcting codes, algorithmics etc.) is intricately linked to prime numbers.

KEYWORDS FOR WEBSITES:

Prime Numbers - Divisibility criteria - Eratosthenes - Euclid Algorithm Cryptography - Public-key Encryption



To notice

After counting with small pebbles (calculus in Latin), humans invented the decimal notation system. Therefore, saying there are 549 sheeps means that there are actually $5 \times 10^2 + 4 \times 10 + 9$ sheeps. This system uses ten symbols (the figures) and takes into account the position of each of them (from right to left). Nowadays, using mathematical logics, base-2 notation system enables representing situations when only two statuses are possible: true or false, yes or no, the current flows or not, the magnet is magnetized or not, the light is highly reflected or hardly... Digital technology is now everywhere in our lives. This technology is able to reproduce reality with ever-increasing quality. The elementary and basic aspect of the binary system is compensated by the tremendous amount of data processed in a very short time by this technology.

To notice

A color digital picture is made of unit squares called pixels and each square is a mix of 3 or 4 colours: red, green and blue (RGB) for screens and cyan, magenta, yellow and black (CMYB) for colour printing industry. For a given surface, the more pixels used, the higher the definition..

Do it yourself

MATERIAL: 1 paper, 1 pen

1 + 1 = 10 !!!

In a base-2 system, integers are written with only 0 and 1: 0, 1, 10, 11, 100, 101, 110... Using the base-2 system, calculate: $101 + 11$, $101 + 101$, $111 + 111$... Then: 101×11 , 101×101 , 111×111

$$\begin{array}{r} \text{||} \\ + \text{||} \\ \hline \text{||0} \end{array} \quad \begin{array}{r} \text{||} \\ \times \text{||} \\ \hline \text{||} \\ \hline \text{||01} \end{array}$$

Do it yourself

MATERIAL: 3 digitized pictures

72 dpi or 300 dpi?

These pictures are the same size: 7 x 4.35 cm.

- How many squares (pixels) are they composed of (width x height)?

This number corresponds to the size of the picture in pixels.

Picture 1 has a 75 dpi resolution (dot per inch), picture 2: 150 dpi, picture 3: 300 dpi.

- How does the number of squares vary according to the resolution?

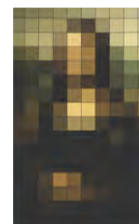


image 1

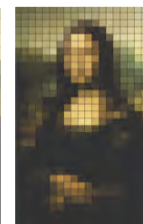


image 2

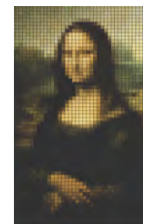


image 3

The size of pictures

How to change the size of a picture? Fill in the opposite table.

A 75 dpi digitized picture is here presented with different resolutions: 75, 150 and 300 dpi.

- The first resolution sufficient for a screen, produces a pixelated printed image.
- The 2d one is sufficient for tablets and current smartphones, but the printed image lacks neatness.
- The third one, 4 times finer, is used to get a clear and smooth photographic printing.



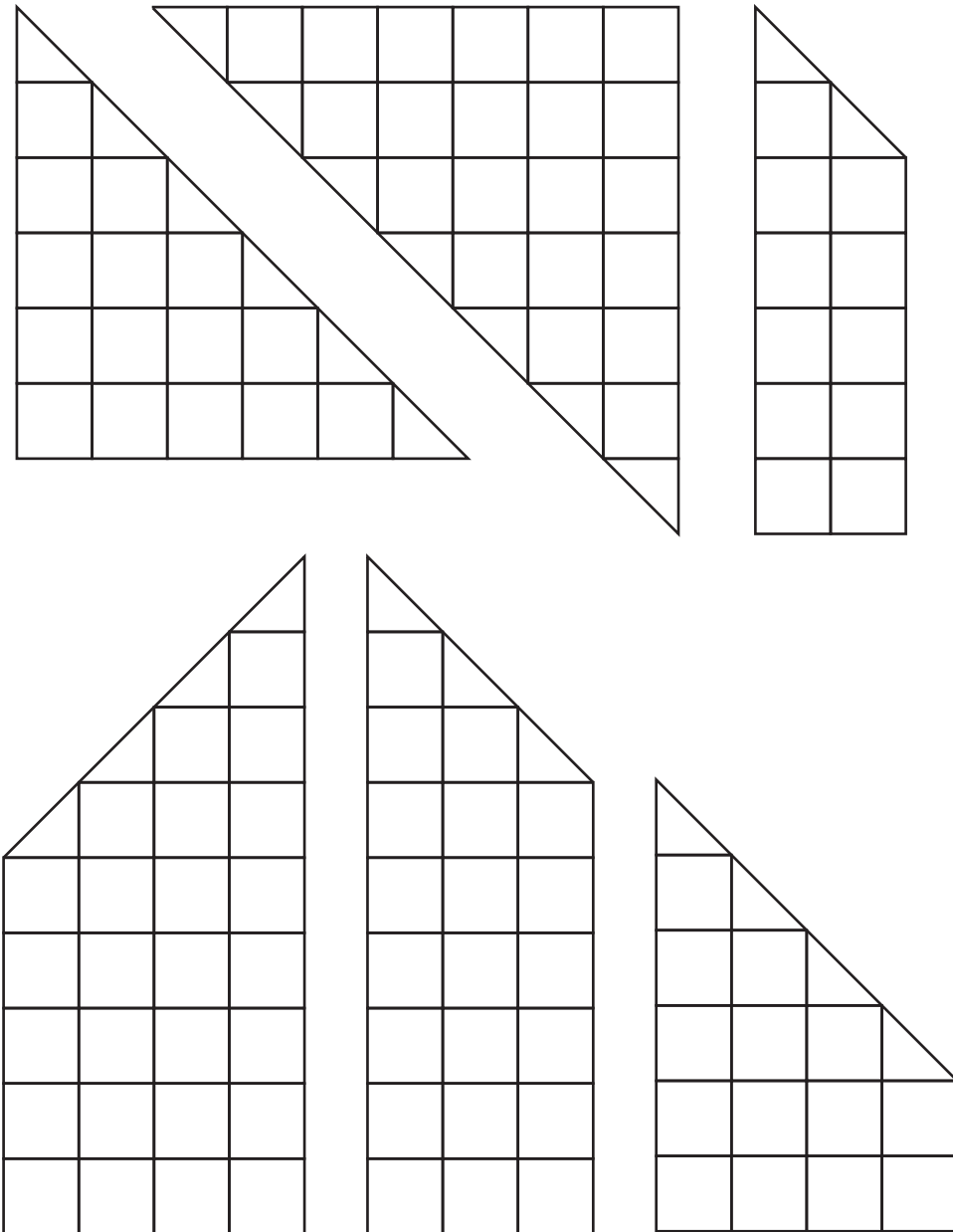
Picture size		Resolution in dpi	Size in K octets
pouces	cm		
3,74x2,39	8,74x5,39	75	125 Ko
		100	?
		150	?
		300	?
1,90x1,18	4,45x2,75	75	?
		100	?
		150	125 Ko
		300	?
0,94x0,59	2,20x1,39	75	?
		100	?
		150	?
		300	125 Ko

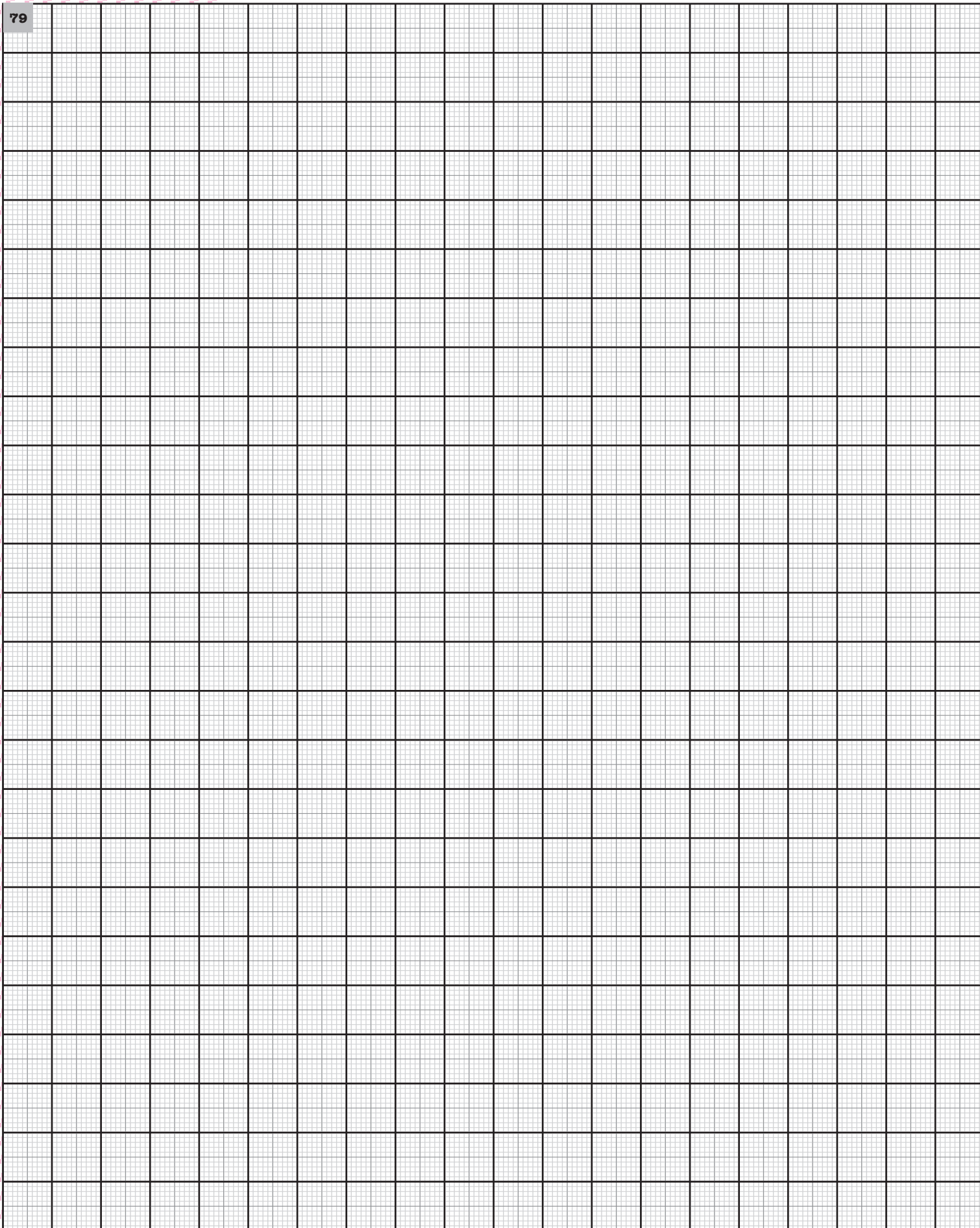
Maths-related jobs

Photographs, CDs, DVDs, the Internet, mobile phones, high definition TV sets... all of them use digitized pictures. Mathematics has become essential for the storing, the processing, analysis, transmission, but also for the compression, correction, modification of pictures.

KEYWORDS FOR WEBSITES:

Pixels - Image file compression - Digital printing





1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102
103	104	105	106	107	108
109	110	111	112	113	114
115	116	117	118	119	120
121	122	123	124	125	126
127	128	129	130	131	132
133	134	135	136	137	138
139	140	141	142	143	144

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
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141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160
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6. Constructing



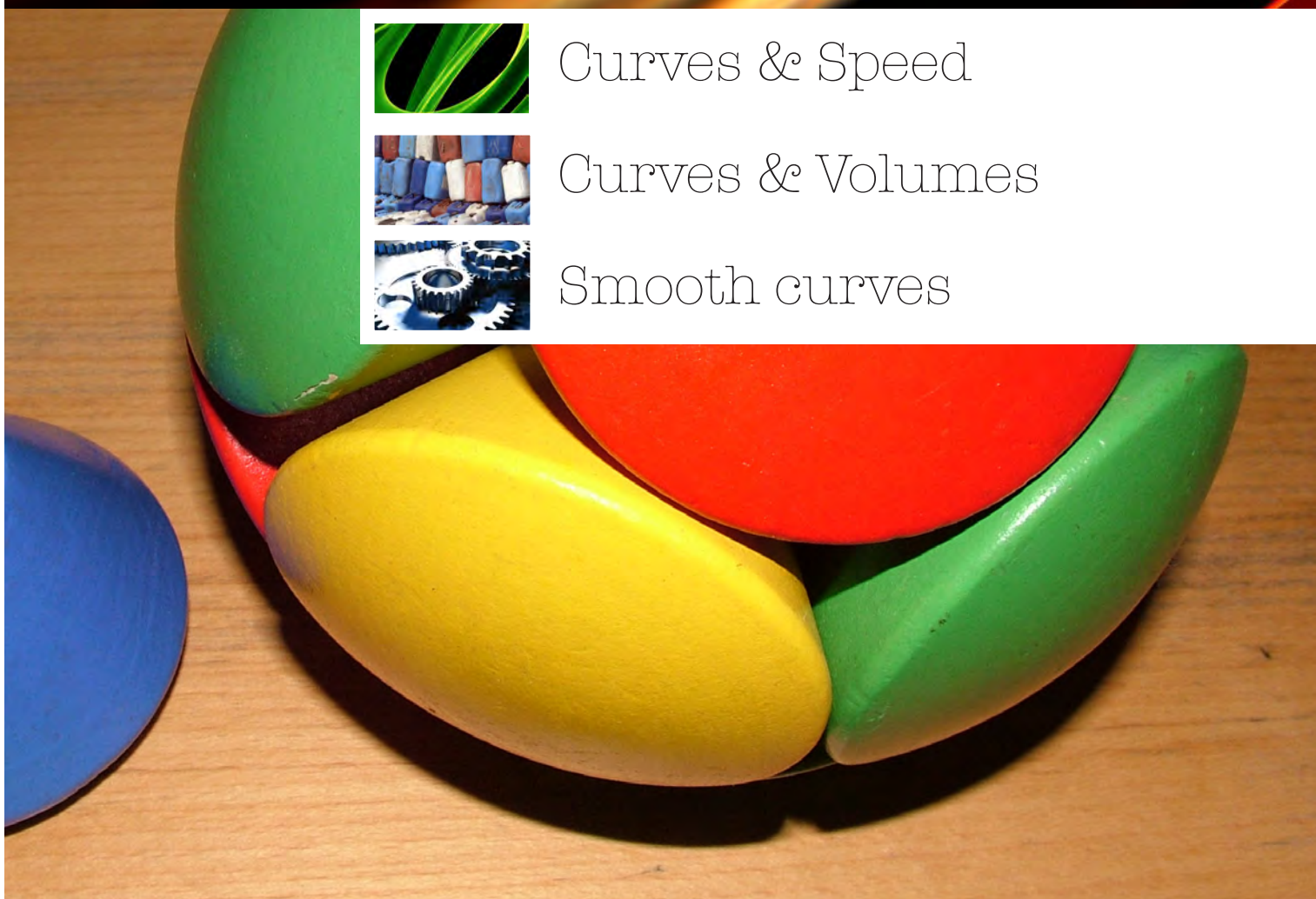
Curves & Speed



Curves & Volumes



Smooth curves



Curves & Speed



Do it yourself

MATERIAL: 5 race circuits designs, 1 speed-time graph

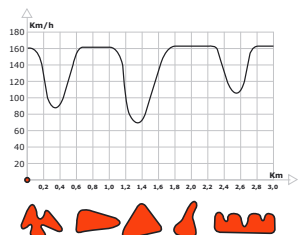
88

To Remember

Speed and Acceleration

Speed is a physical measurement which enables to know and describe the rate of change of a quantity (distance in most cases) with time. It may be expressed in meters per second, in km per hour, in knots (nautical miles) and in Machs for aircrafts.

Acceleration is the rate of change of the speed of an object as a function of time. The function «speed/time» is the derivative of the function «distance/time». Similarly, the function «acceleration» is the **derivative** of the function «speed». It describes the rate of change of the tangent with respect to each point of the speed/time curve.



Which race circuit are you on?

The graph shows the speed of a race car at full capacity in its third lap.

Questions:

- Which track is the car racing on?
- Which direction is it going?
- Where is the starting line?

Do it yourself

MATERIAL: 1 grid paper, 1 pen

89

Can you read a graph?

- 2 cyclists are riding along the same road. The first one rides at 60 km/h for the first half and at 40 km/h for the way back. The second one rides at a constant speed of 50 km/h for both the way out and back. Which one arrives the first?
- Turn the following story into a graph: A walker climbs a hill 4 km long at a speed of 2 km/h. He stops for one hour and descends the same hill at a speed of 4 km/h.
- Imagine a race circuit and draw it. Then, imagine what the speed graph would be for one lap and draw it too. Exchange graph with the person next to you.

Question: from the graph, try to find out which circuit your schoolmate/partner imagined. The first person to find the solution

Going further...

Lower your speed

- A car drives at an average speed of 100 km/h instead of 90 km/h. For a 90 km distance, how much time does it gain?
- A moped drives at a speed of 50 km/h instead of 45 km/h. For a 45 km distance, what is the time gain? For a 90 km distance?

Brake in time!

The **reaction time** between the moment a driver perceives an obstacle and the moment the driver starts braking is estimated to be between 1 and 2 seconds.

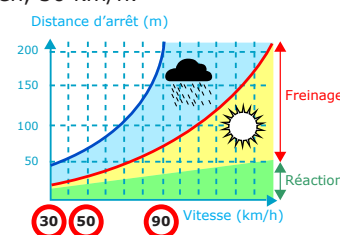
The **braking distance** varies according to the speed and the state of the vehicle but is usually considered to be $0.08 v^2$ where v is the speed of the vehicle expressed in meter/second.

- Calculate the distance it takes for a car to stop when it is going 90 km/h and 100 km/h.
- Perform the same calculation for the moped going 45 km/h, then, 50 km/h.

Wet road conditions

The condition of the road also needs to be taken into account as, for instance, on wet roads the braking distance is increased by 40%.

- Calculate the previous braking distances for wet road conditions.
- Create a graph with the various braking distances taking into account both speed and road condition.



Maths-related jobs

The **differential calculus** was founded in the 17th and 18th centuries. It enables describing the rate of change of the slope of the tangent to a continuous curve.

- If the function is rising, the tangents to the curve at each point have a positive slope.
- If the function is falling, then, the slope is negative.

This property is most useful for people who study evolutive phenomena such as mathematicians, physicists, but also engineers, biologists, demographers, economists...

KEYWORDS FOR WEBSITES:

Function - Graph - Derivative - Tangent - Speed - Acceleration

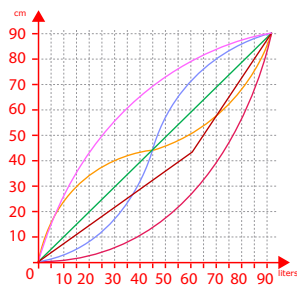


Do it yourself

MATERIAL: 1 drawing of 6 containers, 1 drawing with 6 graphs on it

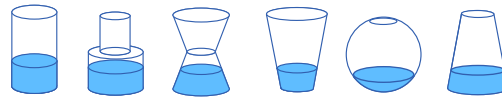
90

91



The volume of the barrel

- The six containers have the same height (90 cm) and the same volume (90 l). The graph shows the fill level of the containers as a function of time.
- Match each curve to the corresponding container.
- Can you guess what the dimensions of the containers are?



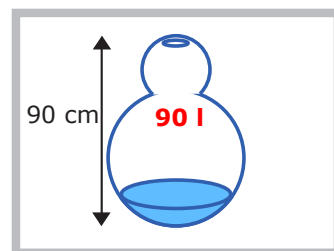
Do it yourself

MATERIAL: 1 grid paper, 1 pen

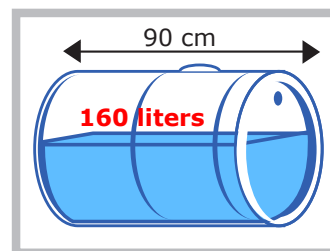
89

Can you read a graph?

- Each of you imagines and draws a container (90 cm - 90 l).
- Then, you build the graph corresponding to fill level of your container and pass it to the person next to you.

Question: the first person to find wins!

- What is the fill-level curve for this container?



- What is the fill-level curve for this barrel?

Do it yourself

MATERIAL: a drawing of the 6 containers, a drawing of 6 graduated gauges

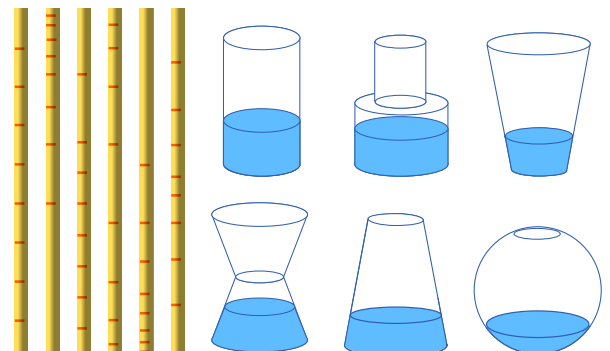
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One gauge each

- Each stick is the graduated gauge of one of the containers. Knowing that the interval between two marks is 10 liters, which gauge goes with which container?
- Graduate the gauge with 10 liters intervals of this 160 liters lying barrel with respect to the position of the hole (whether it is on the side of the barrel or on the lid)

Empty the tanks

- Go back to the 6 containers and the fill-level graph again
- Now, assuming that the flow is constant and that the emptying plug is located at the bottom of the container, draw the corresponding emptying-level graph for each of them.





To notice

To determine the fill-level curve of a container at any given time, you need to estimate (or calculate) the rate of change of the area of the section of the container as a function of height h , which is the same as calculating directly or approximately fluid volume as a function of fluid height. This is what integral calculus, introduced by **Newton** (1642-1727) and **Leibniz** (1646-1716), is about. However, many volume (and area) formulas have been known ever since **Archimedes** (287-212 B.C)

Interesting Formulas

Volumes of containers as a function of height
(height in dm, volume in dm³ or liters)

- Cylinder: $V(h) = \text{Base} \times h = \pi R^2 \times h$
- Pyramid on basis: $V(h) = B \times h/3$
- Cone on point: $V(h) = V(H) \times (h/H)^3$
- Sphere: $V(h) = \pi h^2 (R - h/3)$

Maths-related jobs

In mathematics, the notion of relation between two (or among several) variables is expressed in **functions**. Their **graphical representation** are part of our everyday lives (temperature graphs, stock exchange graphs...) and used as tools in many technical fields.

Like differential calculus, **integral calculus**, which enables calculating areas and volumes, was developed in the 17th century. Solving volume and gauge problems has always been part of merchants and engineers' tasks.

KEYWORDS FOR WEBSITES:

Function - Graph - Integral calculus - Volumes - Archimedes - Barrel



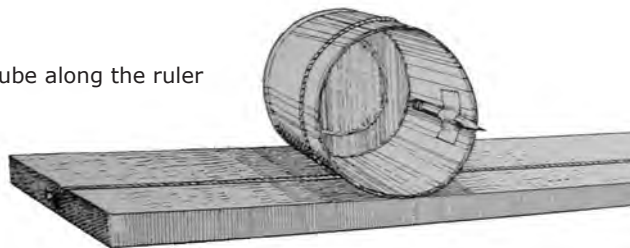
Do it yourself

MATERIAL: 1 pen, 1 paper, 1 ruler, 1 tube

Roulette and cycloid

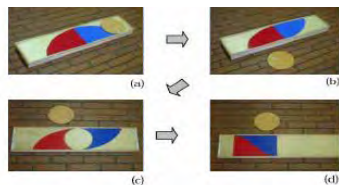
- Fix the pen inside the tube.
- Trace the line obtained by rolling the tube along the ruler without slipping.

The curve obtained is called "**cycloid**". It is similar to that of a point on the wheel of a moving bicycle.



To notice

The term «**cycloid**» was first introduced by **Galileo** (1667-1748). This curve has peculiar properties. Two balls being dropped from different heights will meet always at the bottom of the slide. The area located under the branch of a cycloid is three times larger than the area of the disk used to create the cycloid.



The best slide

Which slide should you choose to have your marble reach the end of the slide first?

This problem was first proposed by **John Bernoulli** in 1696, soon after the invention of calculus. It was solved by John himself, his brother James, Guillaume L'Hopital, Leibniz and Newton. This type of problem resulted in the creation of the **calculus of variations**.

This type of cycloid is called a **Brachistochrone** curve.

Do it yourself

MATERIAL: 1 pen, 1 paper PVC or rigid cardboard disks... 1 hollow disk

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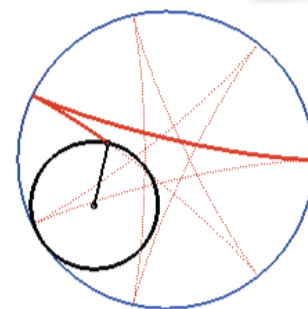
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Lines, curves and loops

- Fix the pen on the outside of the disk.
- Trace curves by rolling this disk inside or outside another disk.

These curves are called **hypocycloids** or **epicycloids**.

- Try to create a similar cycloid with 1, 2, 3... outside branch.
- Try to create a similar cycloid with 1, 2, 3... inside branch.



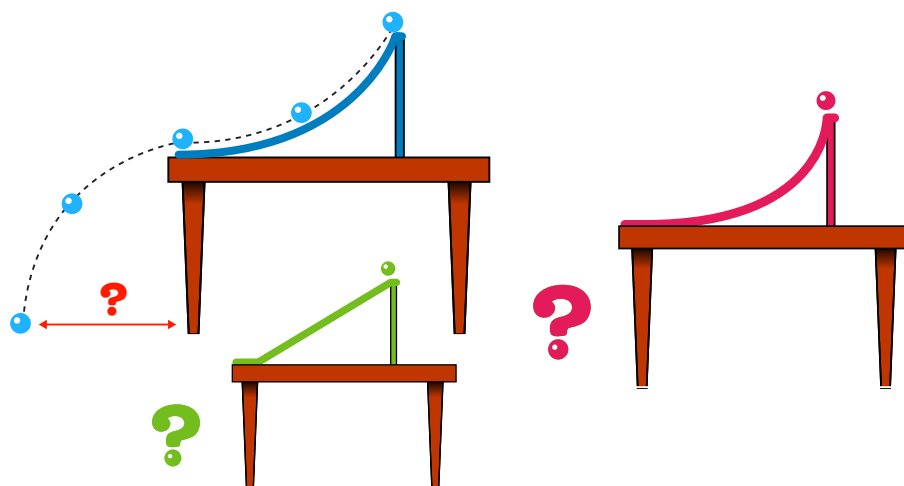
Do it yourself

MATERIAL: 3 wooden or U-shaped PVC slides, 3 identical marbles

The shortest path

Place the slides on the edge of a table. Drop the three balls at the same time.

- Which ball will reach the end of the slide first?
- Once on the floor, which of them will roll the furthest?



Morale of the story: «a straight line is not always the shortest way»



Going further...

Clashing problems

Some problems have solutions which are against logical or natural reasoning.



Take 2 identical coins.

Make the first one spin around the second one which is completely fixed.

- How many rotations will it make in a round trip?



Take a bottle. Lay it flat on the table and spin it keeping its neck on a piece of wood.

- What distance did the bottle cover in a round trip?
- What about a dot on the bottleneck? Why?



Take a board standing on two identical cylinders.

- What is the distance covered by the plank when the cylinders make a round trip?

Maths-related jobs

Aristotle and **Ptolemaeus** already used these curves to describe the motion of planets. Astronomers still use them today.

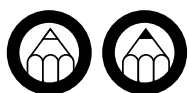
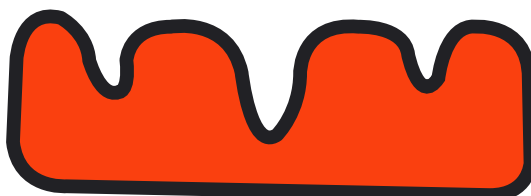
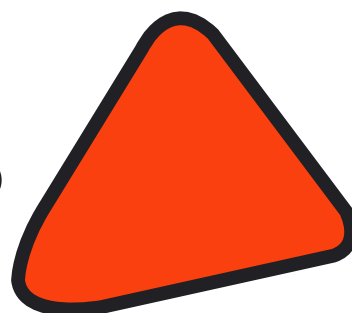
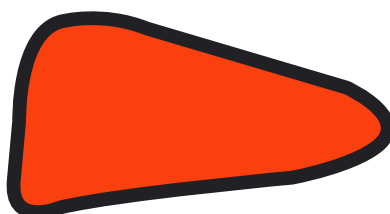
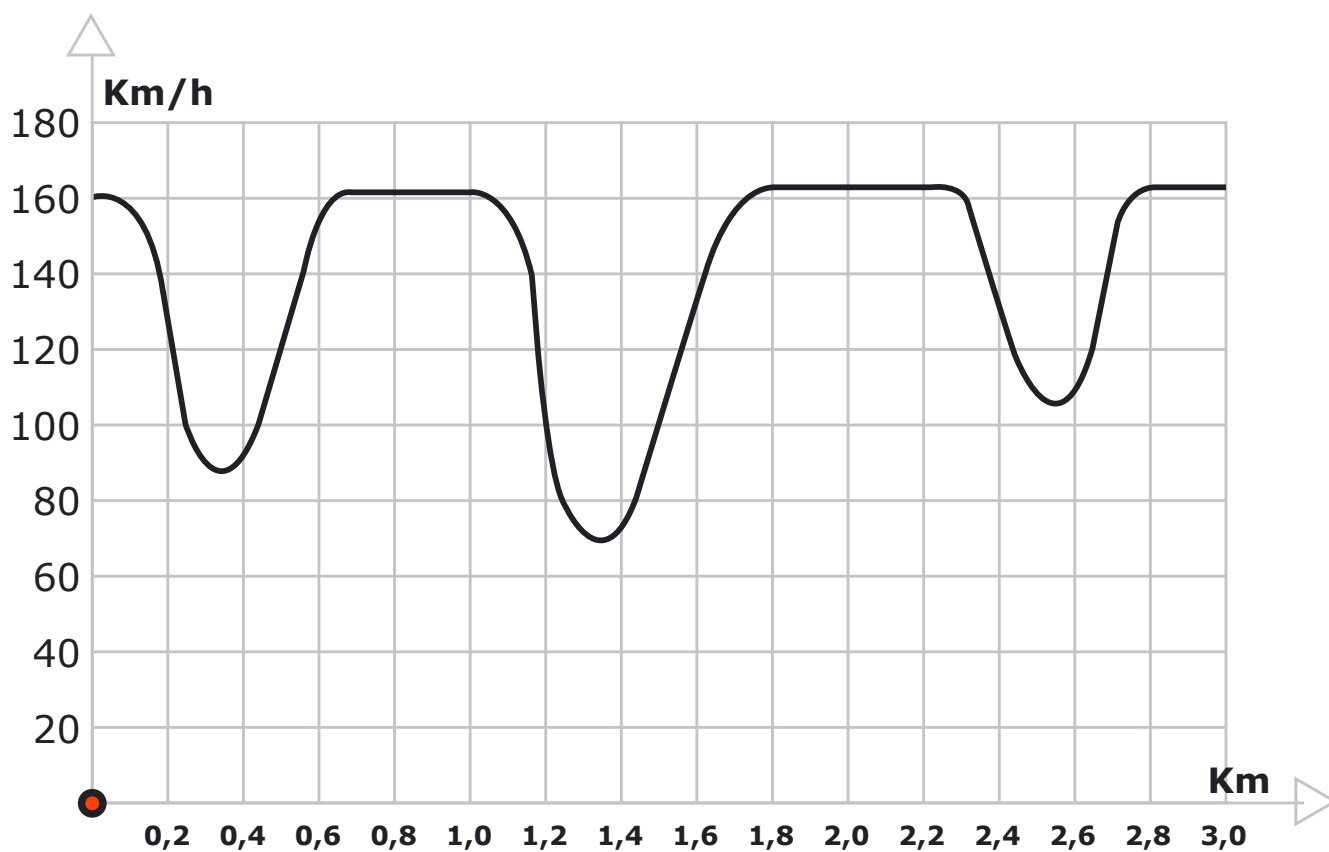
In 1659, while looking for a solution to get more accurate clocks for naval navigation and astronomy, **Huygens** invented a pendulum clock swinging between two cycloidal cheeks. This pendulum is isochronous.

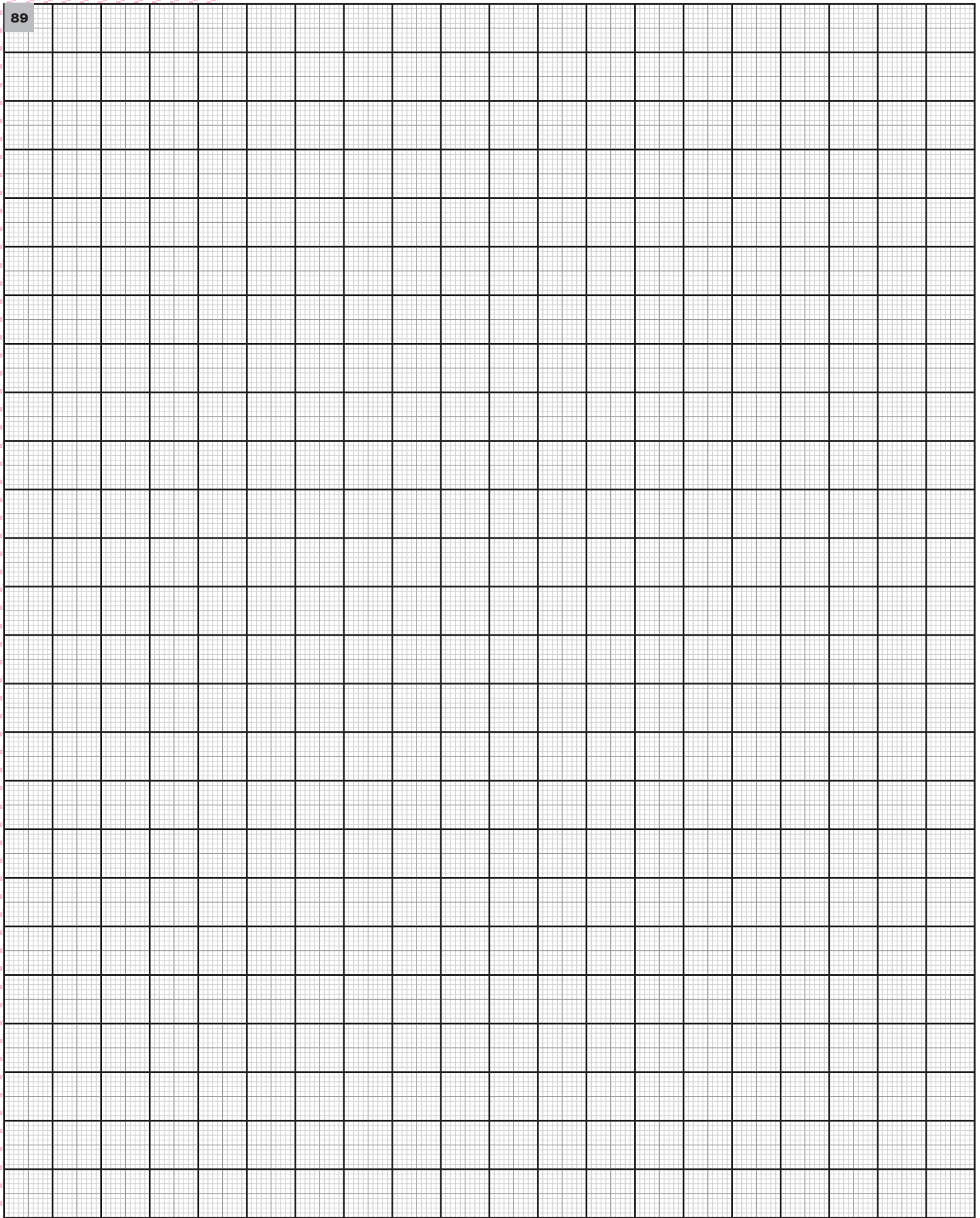
In mechanics, cycloidals are used in gears and speed regulators.

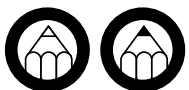
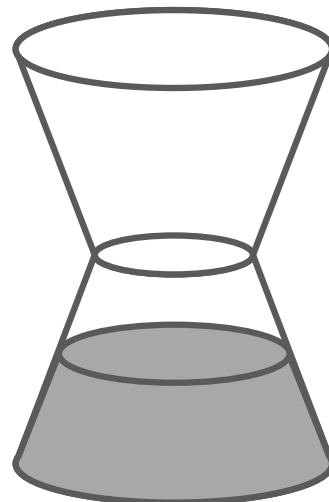
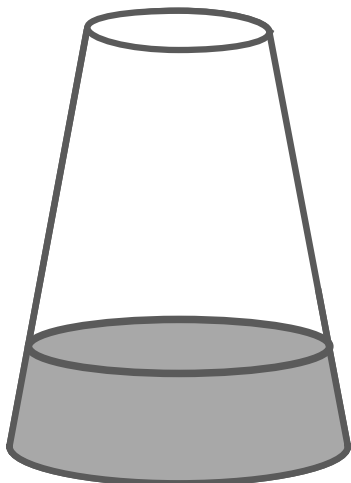
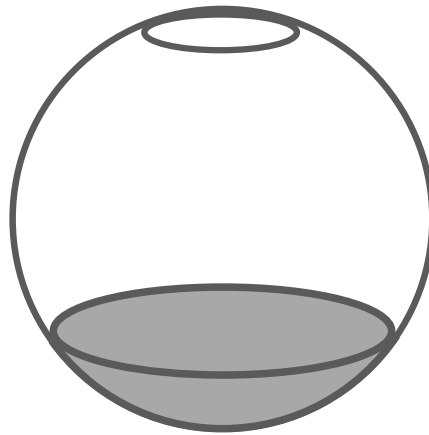
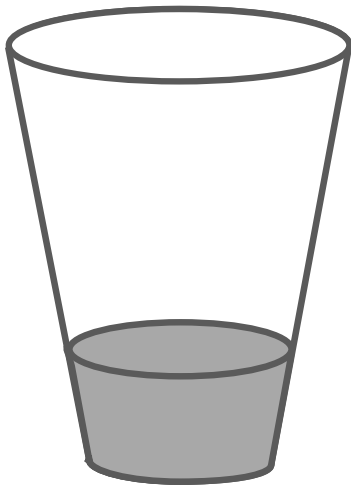
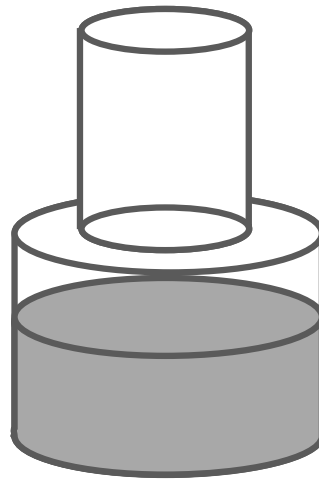
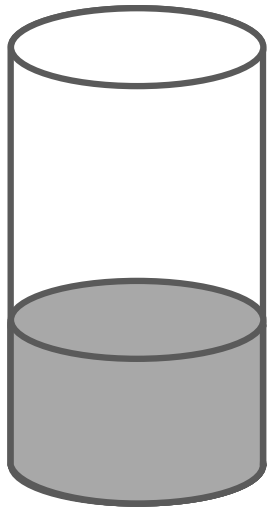
A cycloid skate-board slope would certainly be more interesting than the present ones!

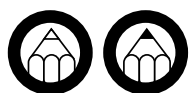
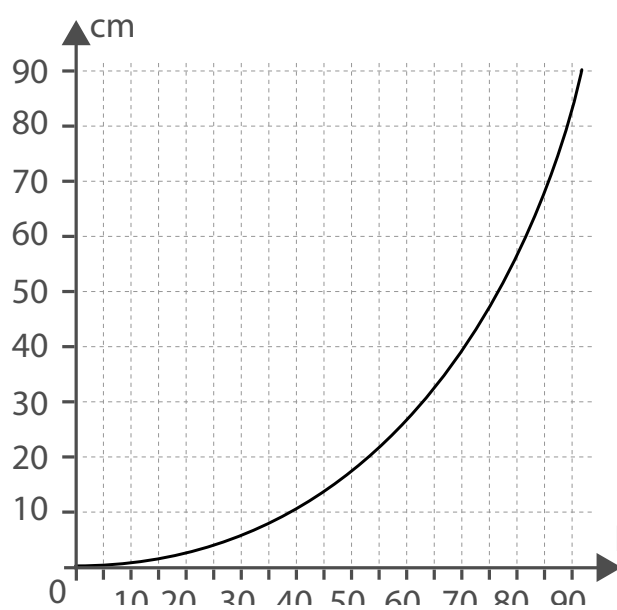
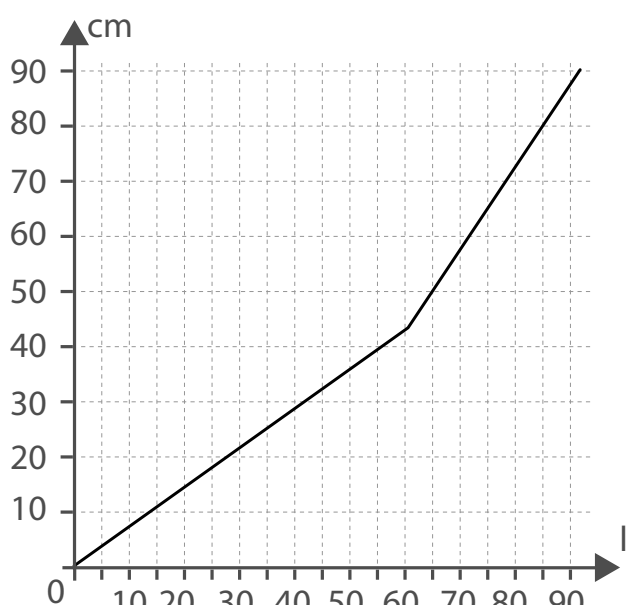
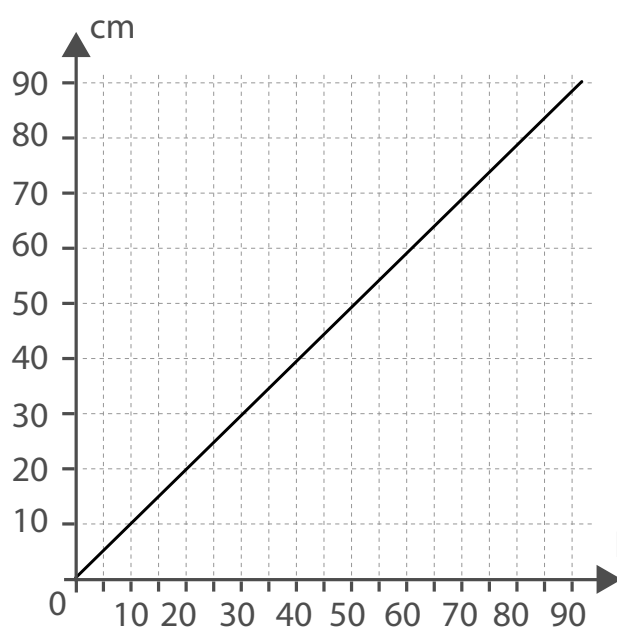
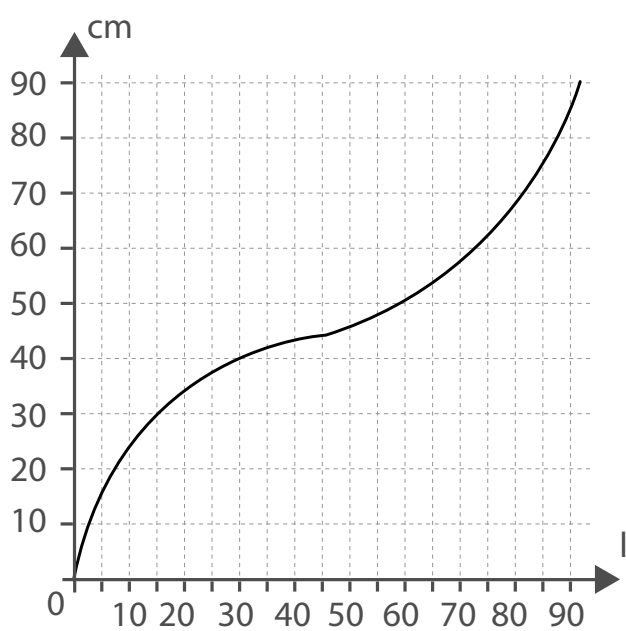
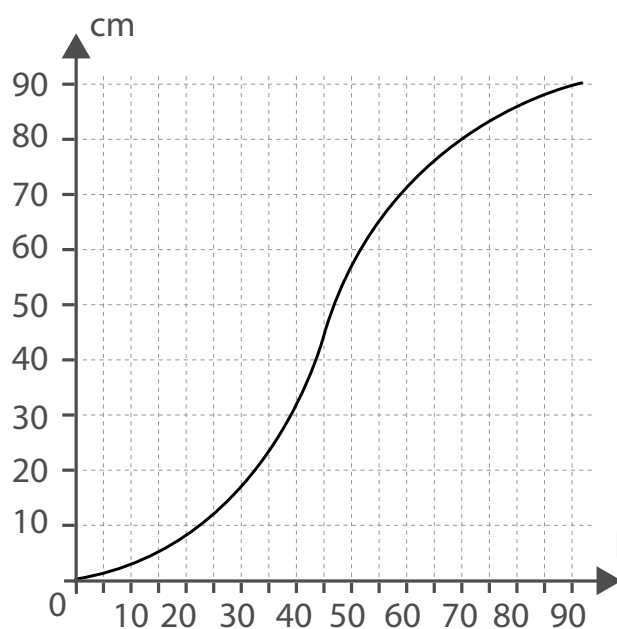
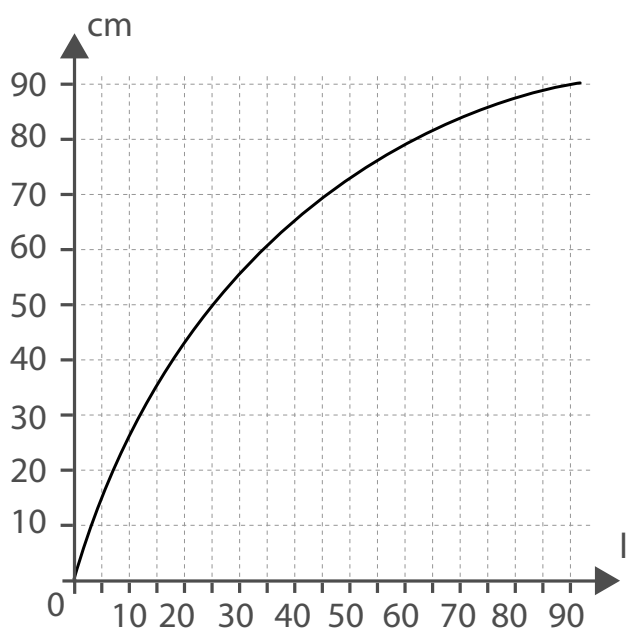
KEYWORDS FOR WEBSITES:

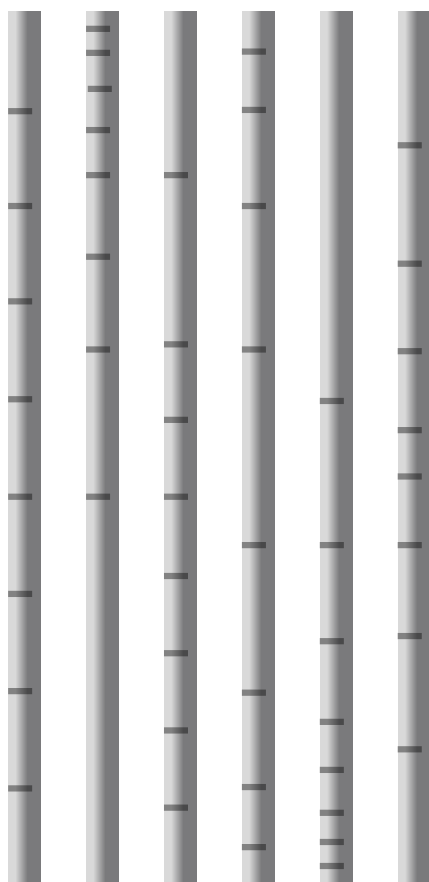
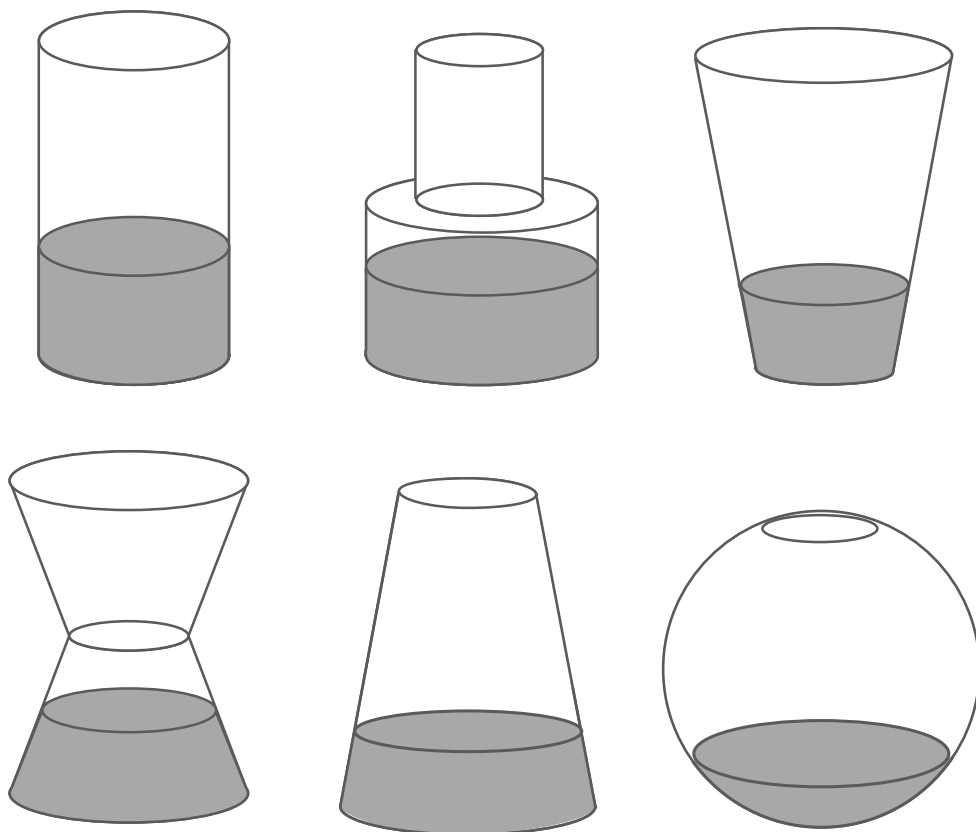
Bernoulli - Brachistochrone curve - Tautochrone curve - Cycloid - Caustic - Huygens' clock

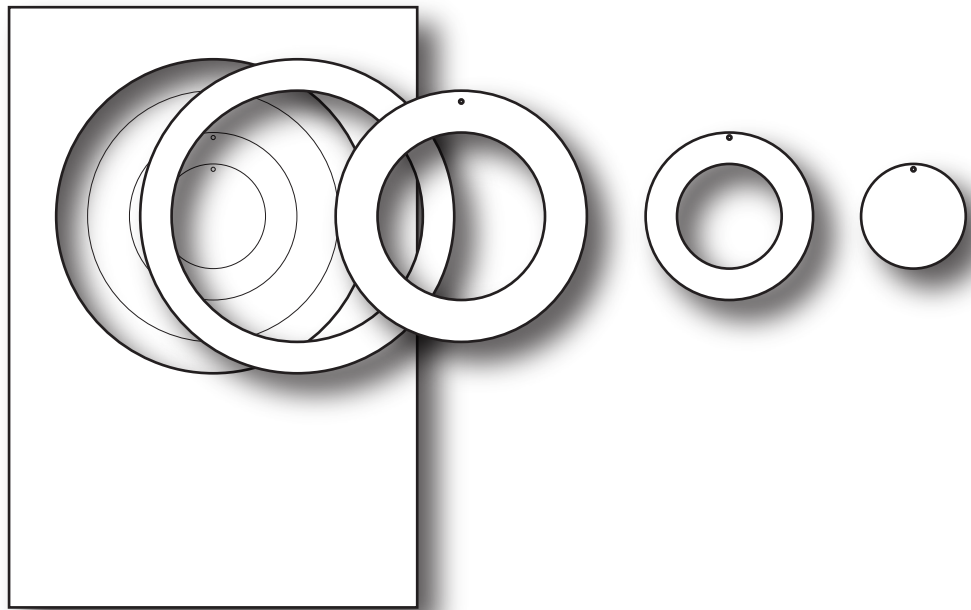
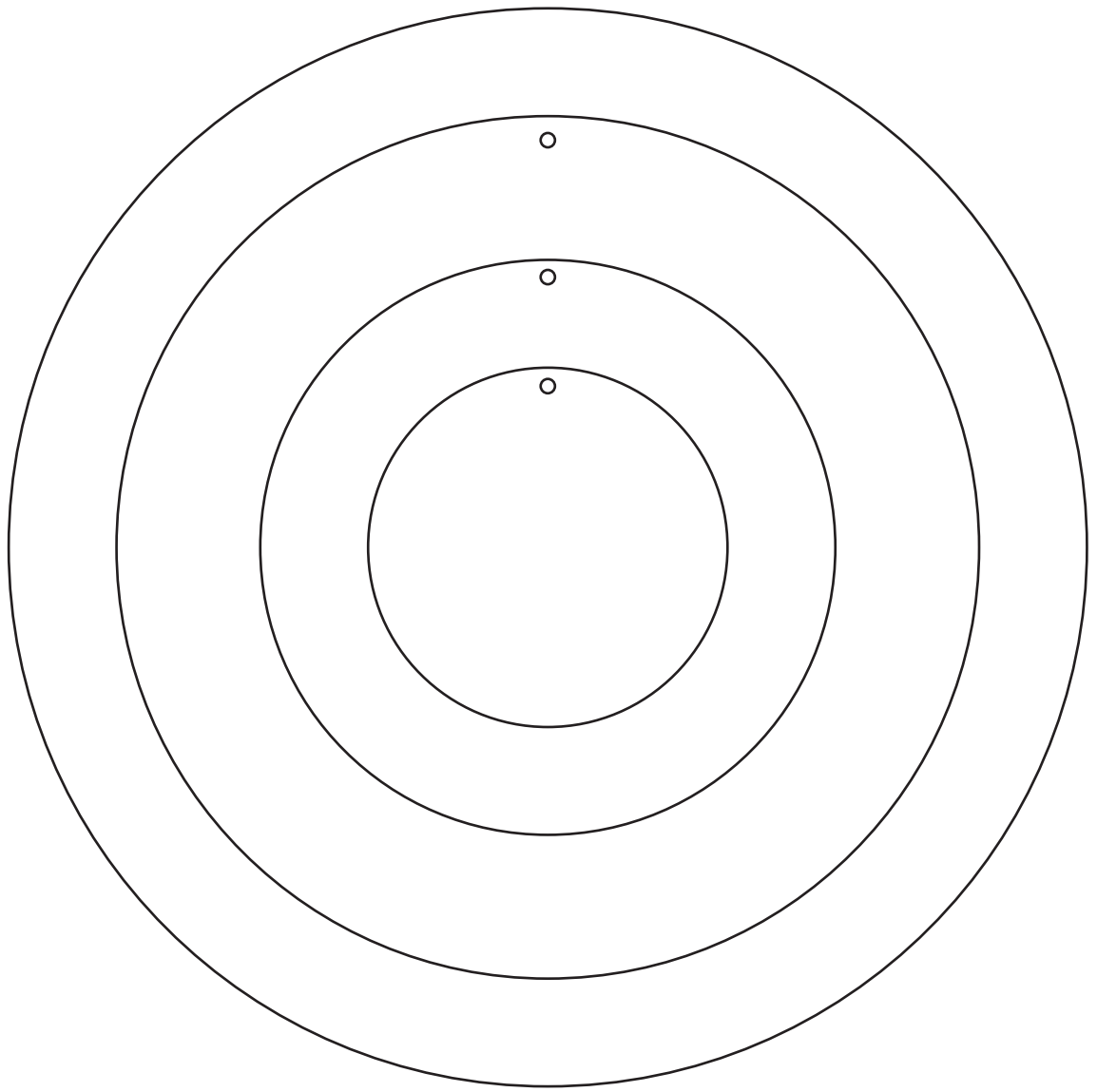


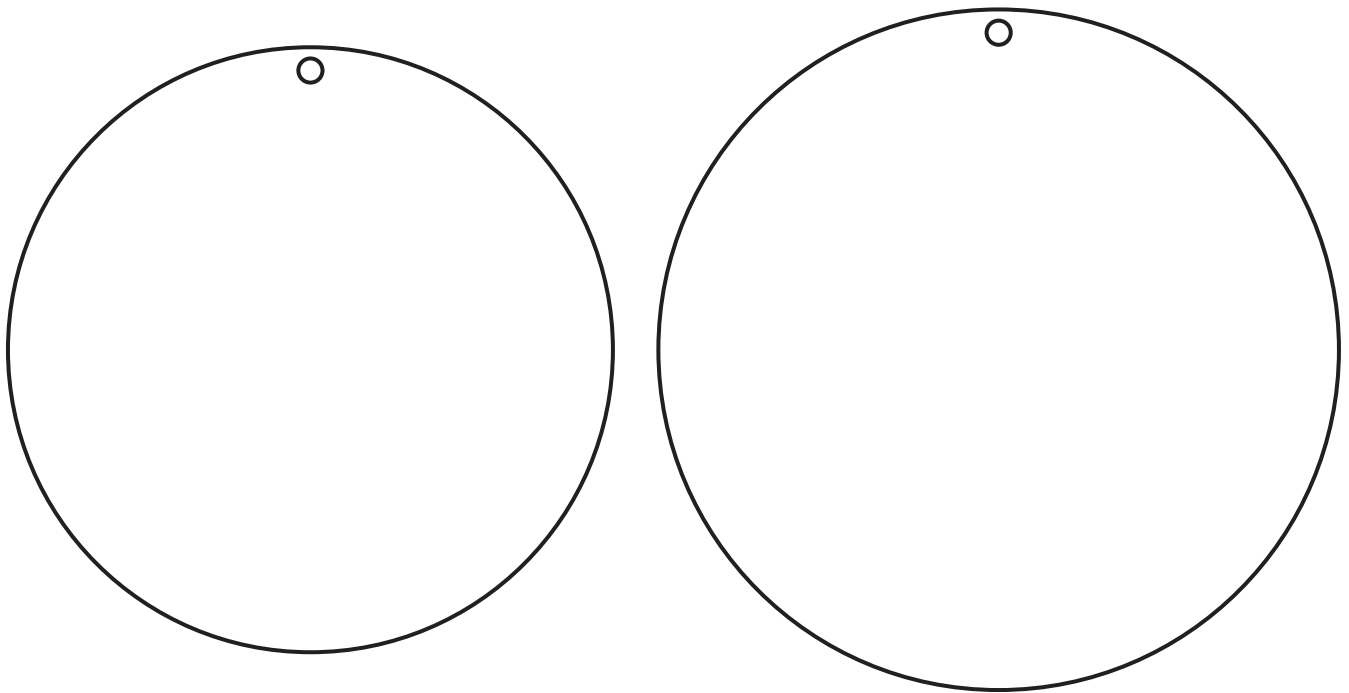
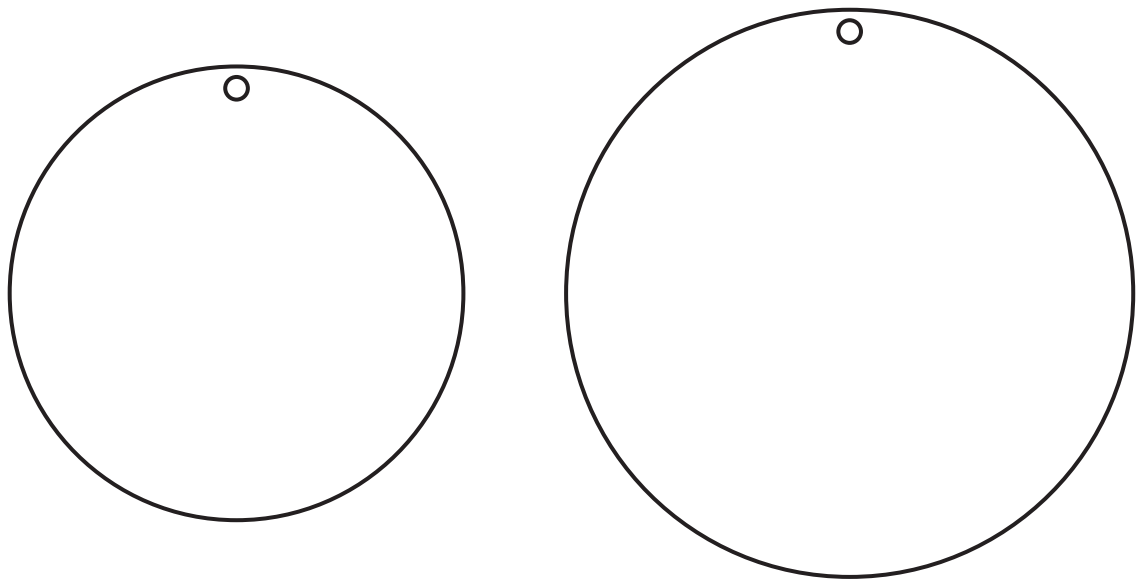
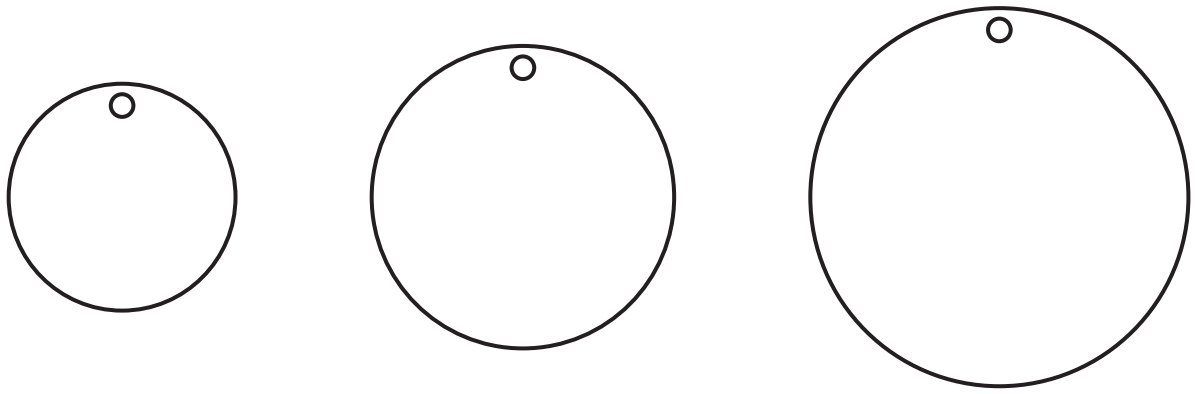










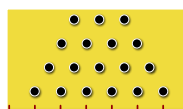




7. Estimating - Predicting



2 red balls?



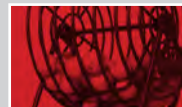
Bingo!



And the winner is?



2 red balls?



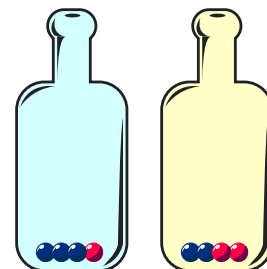
Do it yourself

MATERIAL: 2 bottles with 4 balls

2 balls of the same colour or 2 balls of different colours?

Turn one bottle upside down.

- Do you have a better chance of getting 2 balls of the same colour, or 2 balls of different colours?
- How to be sure?
- What if each bottle had 1000 times more balls of each colour?



To notice

Predicting and estimating

2 balls of each colour are placed in a bottle. One would think that the chances of drawing balls of the same colour are the same as drawing balls of different colours. Well, think again!

To prove this, you can:

- repeat the experiment many times. It's a statistical approach.
- calculate how many ways you have to gather 2 balls among 4.

It's a probabilistic approach.

In the first case, you have a statistical estimation of the result. The more you will repeat the experiment, the closer to reality you will be. In the second case, you have a modelised approach of the problem and a theoretical result.

Do it yourself

MATERIAL: 1 box or 1 urn with 750 blue marbles and 250 red marbles (100 balls appear) or 1 bag with 750 blue marbles and 250 red marbles, 1 container with 100 marbles

Balls and Polls

- Draw 100 marbles. How many marbles of each colour did you draw?
- Repeat the experiment several times. Estimate the number of marbles of each colour in the bag.

To notice

If you ask a random group of 100 people chosen among 1000, you will have similar information. Here, a sample of 100 balls informs rather precisely on the number of balls of each colour contained in the box. The number of red marbles lies in a range between 21 and 29. This is what polls are designed for.

Maths-related jobs

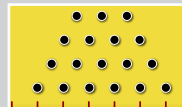
Nowadays, statistics and probabilities are very much used to understand and deal with very complex systems such as rocket launching, errors handling, queues... but also gambling, economics, insurances, pension and pension scheme calculations, quality checks, opinion polls...

Statistics enable extrapolating information collected from questions asked to a small number of people to represent the opinions of a population. Well conducted polls should also inform on the limits of the techniques used.

KEYWORDS FOR WEBSITES:

Chance - Games - Opinion Polls - Luck

Bingo!



Do it yourself

MATERIAL: 1 Galton board (cf plan), marbles, nails, a board

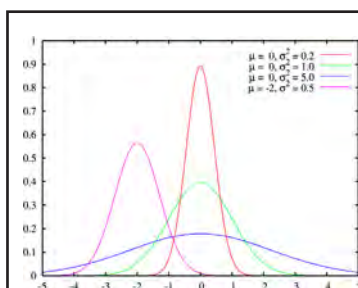
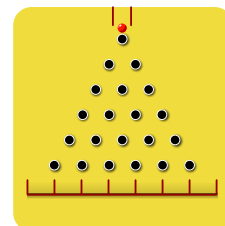
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Is chance calculable?

Choose a marble and drop it gently.

- Can you predict where it will stop?
- Where is it most likely to stop?

Predicting the behaviour and direction of a marble is impossible. However, it is possible to calculate how the whole of the marbles are distributed.



The Gaussian curve

Why is the shape of this curve?
Why is it essential to statistics?

If we sort the inhabitants of a town or a country, according to a characteristic (size, weight, IQ, level of competence...) the closer to the average, the more individuals there are. The further from the average, the fewer they are. At the extremities, there is almost no one.

The graphic representation of this fact is called a belle curve or Gaussian curve - **Gauss** (1777-1855). The universal character of the curve was originated by **Euler** (1707-1783) and **Laplace** (1749-1827) who said that Gaussian distribution is the accumulation of many small and independent factors.

To notice

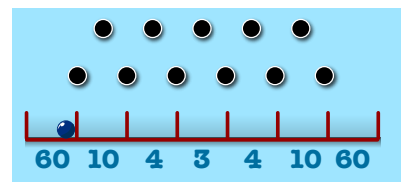
Do it yourself

MATERIAL: 1 Galton board (cf plan), 7 values, 1 under each case

100

Choose the right bin

- Bet one Euro, one dollar or one ... on one of the cases.
- If the marble falls in the case you chose (case), you will win 60, 15, 4 or 3 more times your bet.
- Some cases are more likely to collect marbles than the others. Which ones are they?
- Calculate the number of paths leading to each case.
- For each case, what are the chances of a marble to fall in it?



Simply count the number of paths leading to a bin to know the probability of a marble to fall into it. The numbers found are the same as those of **Pascal's triangle**. The cases do not all have the same chances of winning! Is the owner of this type of game the person who wins most of the time?

Maths-related jobs

Probabilities and statistics are tools used for data and information management. They are also found in information technologies (statistical techniques in signal and picture management), but also in risk assessment (insurance), quality checks, economics, health, financial engineering (computational finances), marketing strategies (market and feasibility study reports...)

Companies or businesses also use Gaussian distribution models for sales and stock management.

KEYWORDS FOR WEBSITES:

Gaussian curve - Statistics - Probabilities - Distribution - Expected gain

And the winner is?



Do it yourself

MATERIAL: 2 players

Rock, Paper, Scissors...



The players count to three aloud (or say «rock, paper, scissors» at the same time) and on the third count show their hands in the shape of their choice.

Paper covers rock, rock smashes scissors and scissors cut paper.

- What is the best tactic to win this game as often as possible?
Answer: Play each turn randomly!
- But, is it really possible to play randomly?

Variation: Add a well! Rock and scissors fall into the well while paper covers it and wins.



Do it yourself

MATERIAL: 2 players then 2 groups of players, black slate, chalk

The race to 20

The first player says 1 or 2. Each one in turn, adds 1 or 2 to the number of the latest player. Write the successive numbers on the board. The first player to reach '20' wins!

- **1st phase:** 1 Vs 1 games.
- **2nd phase:** 1 group of students Vs 1 another group. Allow reflection time between each turn. Students play in turn within the group.
- **3rd phase:** Each group reveals elements of the winning strategy. The other group accepts or refuses the statement. The team with the greatest number of validated statements wins.

Extensions:

- Repeat the race to reach 30 this time. Then, '2010'.
- Play again to 20 or 30 by adding, 1, 2, or 3 at a time.

Do it yourself

MATERIAL: 2 players then 2 groups of players, 1 black slate, 1 chalk, 1 grid

101

Chocolate bars

turn, each player chooses a square of chocolate and crosses all the uncrossed squares located to the left and below of the designated square.



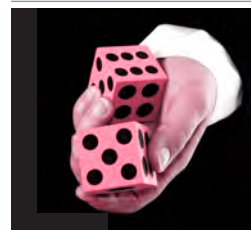
- The loser is the first player to reach the last upper right square.
- Feel free to apply to this game the various rules used for «the race to 20».

Extension:

- Use smaller or bigger grids.

Do it yourself

MATERIAL: 2 players then 2 groups of players, 1 black slate, 1 chalk, 3 dice



Games, chance and strategies

In turn, each player rolls two dice.

- The player with the greatest number wins.
- What are the numbers with the greatest odds?

Extension:

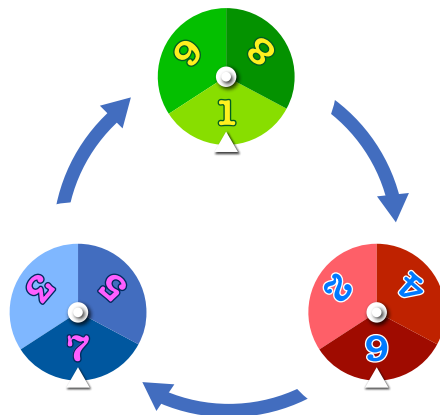
- Play with 3 dice.

And the winner is?



Do it yourself

MATERIAL: 1 player, 3 roulettes



Place your bet!

- Choose a roulette, I will choose the next one and win more often as you!
- Why is that?

Actually, each roulette wins more often than the previous one.

To notice

These games are simple examples of the game theory.

The first two cases are perfect information games with a definite outcome. As the only possible outcomes are «win» or «lose», a winning strategy is crucial. Here, the best strategy is to play first (and well!). However, in the second game, the winning strategy is not exactly obvious!

Thanks to the works of **Blaise Pascal** and **Chevalier de Méré**, the third game helped setting the game theory in the 17th century but it was only developed in the 20th century by **Von Neumann** and **Oskar Morgenstern**.

The last two games show that chance can sometimes be mastered. **Condorcet** (1743-1794) used roulette-like situations to prove that in a democracy, no voting system is better than another.

Maths-related jobs

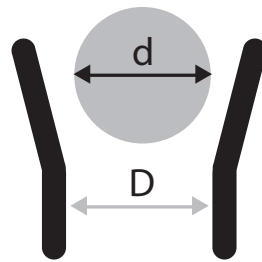
Along with probabilities and statistics, the game theory is very much used in situations requiring strategy, decision-making, competition and cooperation.

The game theory is obviously used for games and gambling, but also in political conflicts (political science), military strategies, economics, business and marketing.

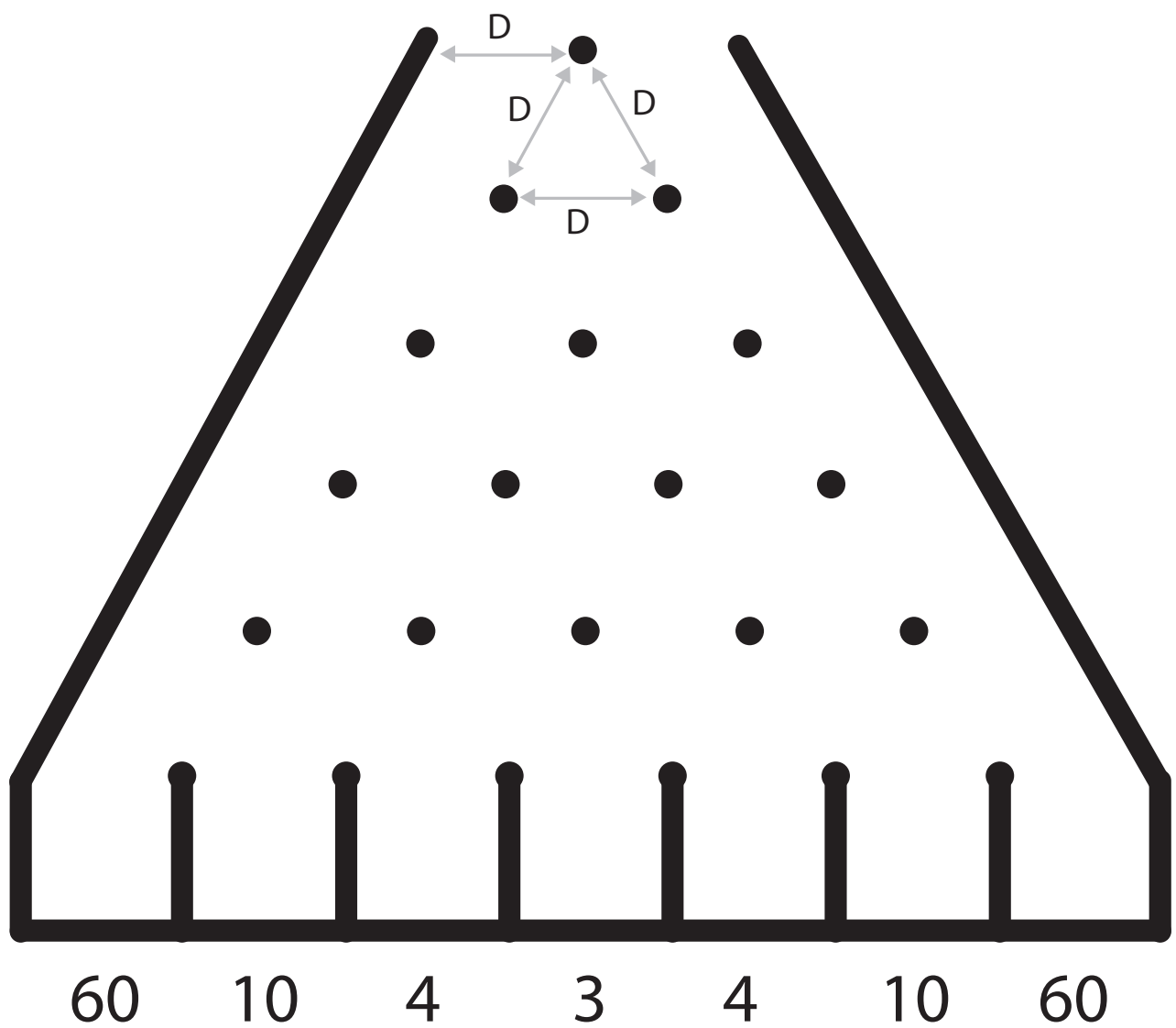
It is also very common in computer science and algorithmics.

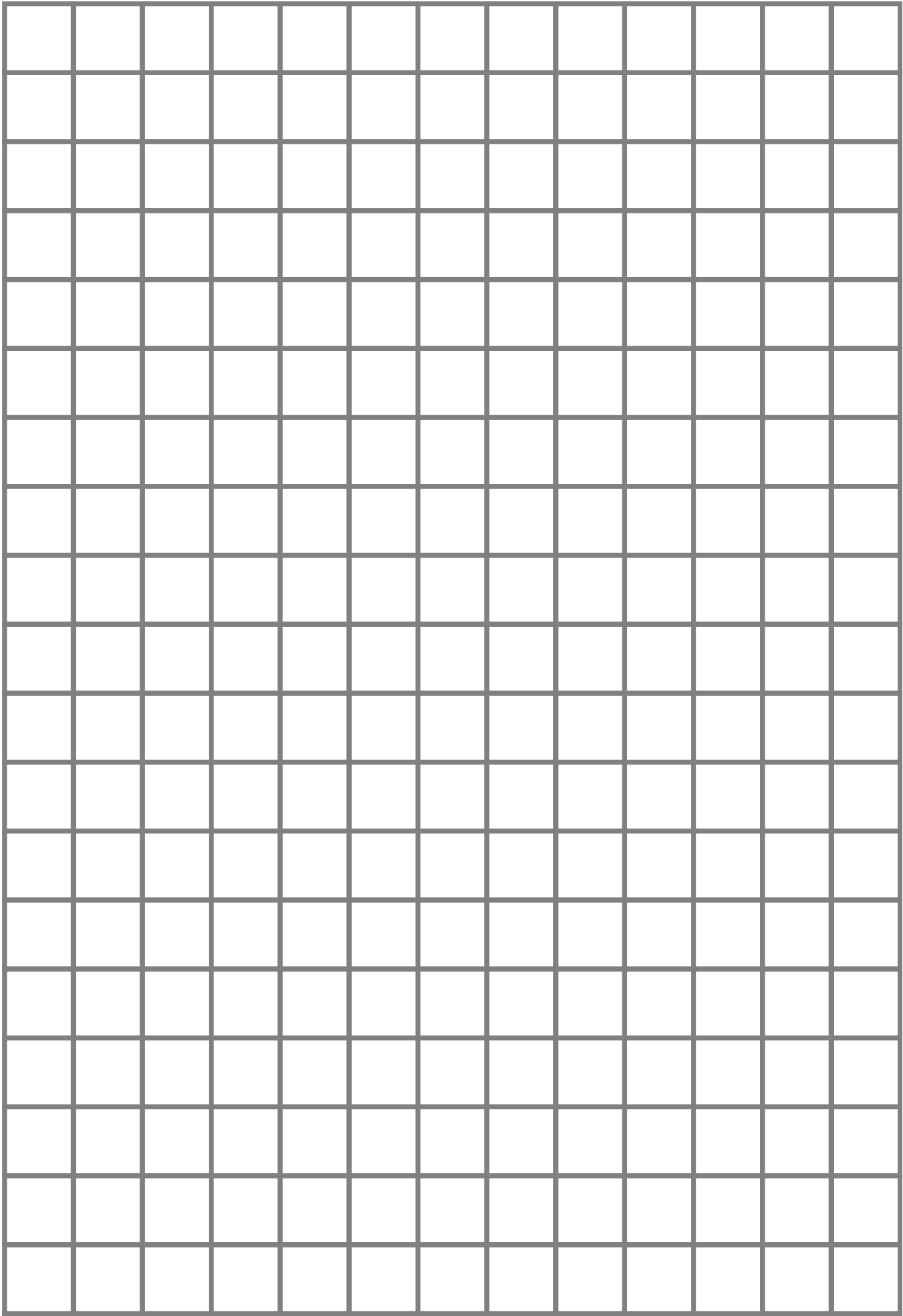
KEYWORDS FOR WEBSITES:

Game theory - Strategy - Von Neumann - Nash - Information theory



$d = 30 \text{ mm}$
 $D = 35 \text{ mm}$





8.Optimisation



Soap bubbles



The shortest path



The best shape

Soap bubbles



Do it yourself

MATERIAL: wire or straws, 1 bucket of soapy water

Lazy Nature

- Build a tetrahedron, a cube, an octahedron, a helix...
- Before plunging the shapes in the water, imagine how the soapy surfaces will spread.
- Plunge the shapes and observe the surfaces.
- How many sides are there around the edges? How many edges around the vertices?



To notice

Soapy mathematics

A soap bubble is spherical. Why?

At a constant area, a circle defines the surface with the smallest perimeter. At a constant volume, the sphere has the smallest surface. Nature tends to go for the easier solution. These shapes offer the least potential energy in proportion with their surfaces.

Soap molecules create a superficial tension which minimizes the surfaces.

In 1860, Belgian physicist **Ferdinand Plateau** was the first to study these shapes and noticed that:

- Bubbles touch surfaces perpendicularly.
- When soapy surfaces meet, they do so:
 - 3 by 3 along a curve at angles of 120° ,
 - 4 by 4 at a point at equal angles ($109^\circ 28' \dots$).

Do it yourself

MATERIAL: Wire, straws, 1 board, soapy water

Bubble to bubble

- Blow a bubble. What shape is it?
- Place a bubble on the board. Describe how the bubble touches the board.
- Place a large bubble and a smaller one on the board, making sure they do not touch each other. Insert a straw between them. What happens?
- Tie a loose string on a U-shaped wire. Plunge it into the water. Pull the string a little. What shape does it take?
- Inside a square wire frame (15 cm side), link the 4 summits with a tied string/ place a tied string (25 cm) inside a square wire frame and tie it to the four summits with a string. Plunge it into the water and see the result. What shape will the inside string take if you pop it with a dry finger?

Maths-related jobs

These shapes are often found in Nature and architecture.

Mathematicians and physicists have been interested in the problems of minimal surfaces for more than 3 centuries, and more recently, over the last fifty years, chemists, biologists, architects...

Honeycombs, skeletons, spider webs... are examples of shapes that Nature optimized throughout their evolutions. Similarly, in car industry, aeronautics, construction, bridge designing... engineers have been looking for optimum solution to lower the weight, energy consumption, dimensions, environmental cost... of the objects they create.

KEYWORDS FOR WEBSITES:

Soap bubble - Minimal surfaces - Superficial tension

The shortest path

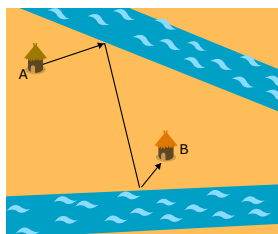


Do it yourself

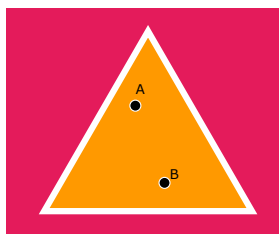
MATERIAL: pen, paper

The shortest path

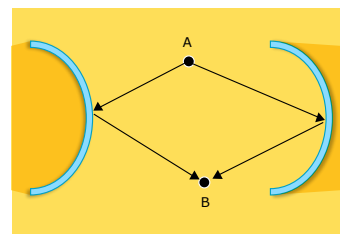
What is the shortest path from A to B?



- Crossing one river?
- Crossing both rivers?



- Following the 3 sides of a triangle?



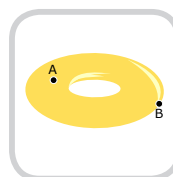
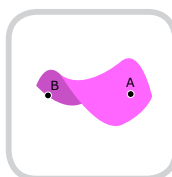
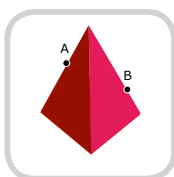
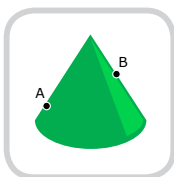
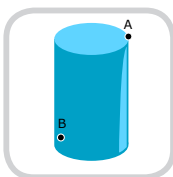
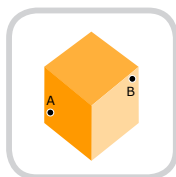
- Using a convex mirror?
- Using a concave mirror?

Do it yourself

MATERIAL: Pen & paper, 1 cube, 1 cylinder, 1 cone, 1 pyramid, 1 «horse saddle», 1 tear

The shortest path on a surface

What is the shortest path from A to B?



To notice

Straight lines are not always the shortest path! The shortest path on a surface is called a **geodesic**.

The most familiar examples are:

- Straight lines **on flat surfaces**, in Euclidean geometry.
- **On developable surfaces** such as cones, cylinders, polyhedra... the shortest path to link two points is also a straight line once the surface has been flattened.
- It is not a straight line anymore when a surface cannot be fully developed and flattened. Saddle-shaped surfaces for instance, have a negative curvature while spheres have a positive curvature.
- The shortest path on a **globe** is a big arc of a circle centred on the centre of the Earth.

Gauss developed these problems in the 19th century.

Do it yourself

MATERIAL: 1 Globe, 1 string or 1 elastic

But what about on Earth?

- Choose two points located around the 40th parallel: Paris and Montreal or Luanda and Sao Paulo...
- What is the shortest path to go from one place to the other?
- Check with the string.
- What does this line on the globe correspond to?



Maths-related jobs

On Earth, issues about finding the shortest path are particularly important for network administrators dealing with fluids, electricity, oil, gas, water but also for terrestrial communications.

Mathematics or algorithmics help solving these problems.

Sailors have been using them for centuries. Nowadays, they are used by pilots, even for long-range flights. When it comes to space flights, these problems become more complex and the gravitational attraction of planets must be taken into account.

KEYWORDS FOR WEBSITES:

Shortest path - Minimum - Curvature - Gauss - Developable surface

The best shape

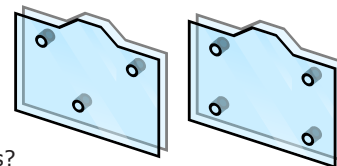


Do it yourself

MATERIAL: 2 acrylic plates connected by 3, 4 or 5 rods, 1 bucket with soapy water

The shortest path between 3 points

Place 3 rods between the acrylic plates*.
Plunge the device in the water. Take it out and watch the result.



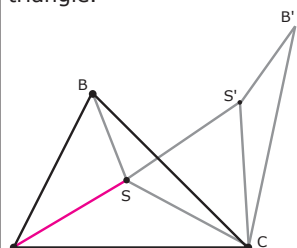
- How many points are there? How many edges? At what angles?
- Now, place 4 rods between the plates.
- Before plunging the device in the bucket, try to imagine how the soap connections will spread: in the shape of an X, a U, a Z, a H?
- Plunge it and observe the number of points, edges.

*What if one of the angles of the triangle is superior to 120°?

To notice

Soapy and mathematical solution

The soapy solution shows us the shortest path to join 3 points, 4 points and more. In the 19th century, Swiss **Joseph Steiner** solved the problem of joining 3 points with a simple solution: Build a triangle CS'B' obtained from a 60° rotation of CSB triangle.



Therefore:

$SA + SB + SC = SA + SS' + S'B'$
This quantity is minimal when the four points A-S-S'-B' are aligned. In this case, angles BSC and CS'B' equal 120°, and the problem is solved!

A school for 3 villages

You may replace the 3, 4 or 5 points by villages, farms... and Steiner's point by a school, hospital... The answer still lies in soap. How can you solve this problem if the number of inhabitants changes?

Do it yourself

MATERIAL: 2 acrylic plates, 1 straw

Honeycombs

- Blow a bubble between the plates. A cylinder will form.
- Blow several small bubbles side by side in the cylinder.
- How do they connect to each other?
- Place a strip perpendicularly to the plates. See what happens.
- If the bubbles are the same size, what shape are they? How do they connect to each other?

To know more

Piling disks leaves empty spaces. Regular hexagons are the shape with the largest surface without gaps. Honeycombs are made of regular hexagons. Have bees found the best solution?

The cell of a honeycomb is not, however, the most efficient shape to occupy a given volume. A better way has since been found, but the best shape is still to be found.

Do it yourself

MATERIAL: 2 wire circles, 4 wire rods

Minimal surface between 2 circles, 2 or 4 straight lines...



- Take the circles out of the bucket and move them away from each other. Where can we find this surface?
- Take two rods out of the bucket, move them away from each other and twist them slightly in opposite directions. Where can we find this surface?
- Repeat the experiment with 4 bars attached to each other.

Maths-related jobs

These minimal surfaces can be found in physicochemistry of materials, in biology, architecture (wing sail structures, concrete...). Materials designed using honeycomb structures have the advantageous properties of being lightweight, strong and rigid. Honeycomb materials made of aluminium are used for the building of the Airbus A380 and the TGV, satellite walls... Paper or polyvinyl honeycomb materials are currently used for doors and pallets...

KEYWORDS FOR WEBSITES:

Optimization - Minimal surfaces - Honeycombs - Stretched structure

9.Proving



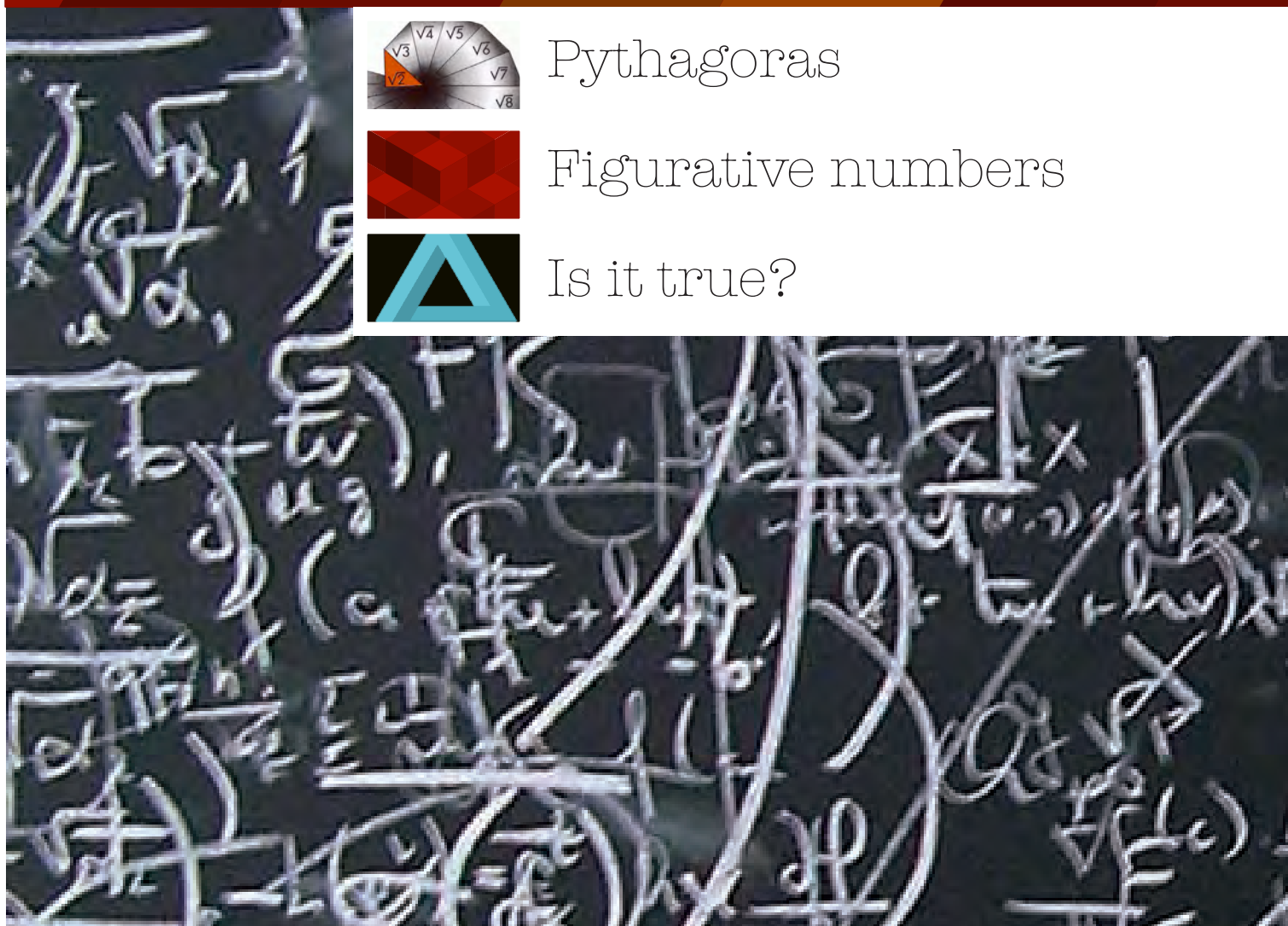
Pythagoras



Figurative numbers



Is it true?



9.Proving Pythagoras



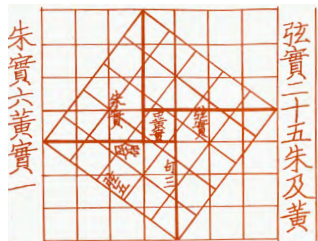
Do it yourself

MATERIAL: 2 cut-out jigsaws

113

114

Que randenir



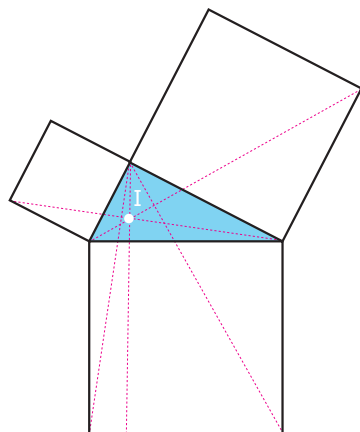
Chou-pei Suan-King
(1105 BC) - China.

3000 years of research

The Ancients, Egyptian or Chinese scholars, already knew many results expressed with integers, such as $3^2 + 4^2 = 5^2$. The Greeks were the first to try and prove this results in a global way.

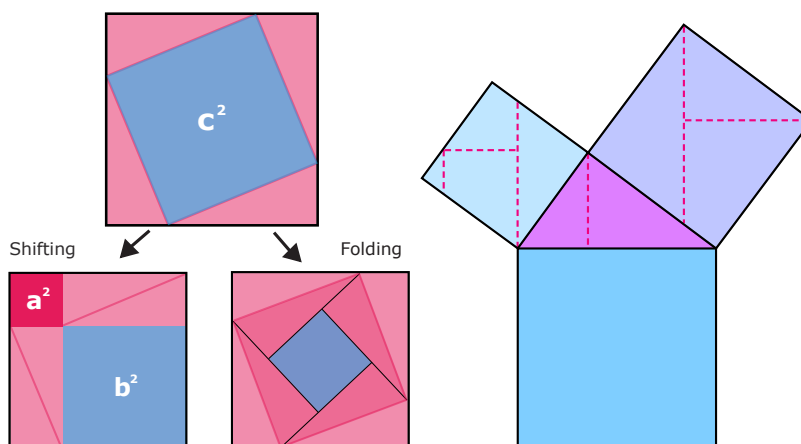
As such, **Pythagoras** (6th century BC) gave his name to the famous theorem «In any right triangle, the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are the two legs». **Euclid** (3rd century BC) gave the most ancient proof known to this date.

There are more than 400 different proofs of the theorem! These proofs opened the way for new problems and new results such as the non-rational nature of real diagonal line length of unit squares.



Pythagorean Theorem

Move and arrange the pieces of each jigsaw to demonstrate Pythagora's theorem, also called Pythagorean theorem.



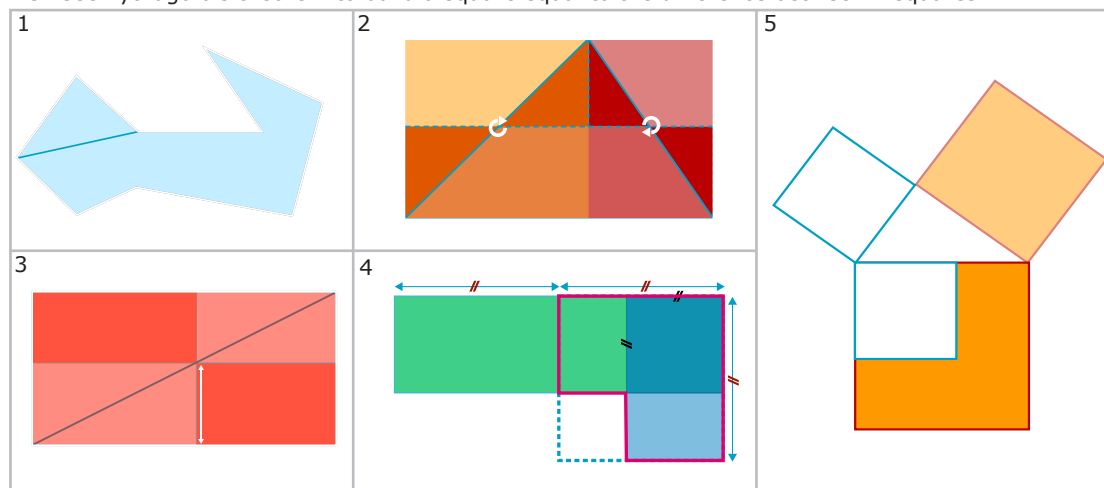
Do it yourself

MATERIAL: Paper, pen, scissors

From polygon to square of the same area

Find the following Euclidean proofs by cutting the shapes out:

- 1- Any polygon can be divided and cut into triangles.
- 2- Any triangle can be divided and cut to reconstruct a rectangle of same area.
- 3- Any rectangle can be divided into another rectangle which width is fixed.
- 4- Any rectangle may be considered as the sum of 2 squares.
- 5- Use Pythagora's theorem to build a square equal to the difference between 2 squares.



Conclusion

Any polygon can be cut out and reconstructed into a square of same area.

Ever since the Greeks, calculating areas is the same as comparing the surface to that of a square. The surface is said to be squarable. Note that it is measured in « **cm², m², km²...** ».

9.Proving Pythagoras



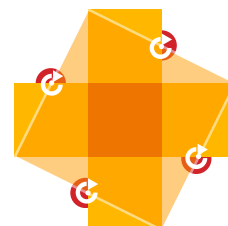
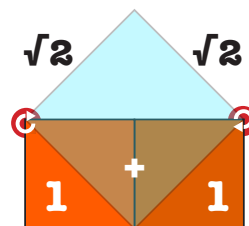
Do it yourself

MATERIAL: Paper, pen, scissors

Duplicate, triplicate...

- Cut the pieces out and build a square,
- equal to the sum of 2 squares,
 - equal to twice the size of a given square,
 - equal to 3 times, 5 times... a given square.

Try to use as little pieces as possible.



To notice

The Chinese considered mathematical and arithmetical problems as geometrical, like jigsaw puzzles. For **the Greeks**, these squaring and duplication problems led to famous problems:

- the irrationality of the diagonal length of unit squares,
- the duplication of cubes,
- the squaring of circles..

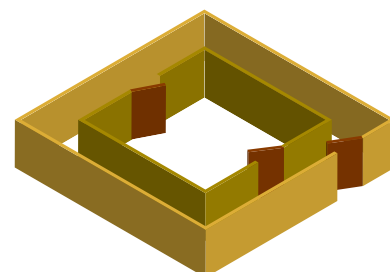
Do it yourself

MATERIAL: Paper, 1 pen

The sofa

- For a sofa to pass through a 1-meter wide corridor with a right angle, what are the largest dimensions it can have?
- Can it go into the main room?
- What if it the sofa was a simple plank (or ladder)?

NB: the dimensions of the doors are: 80 x 210 cm!



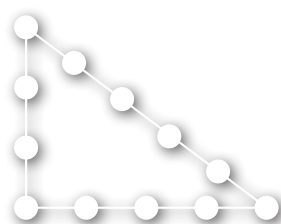
Maths-related jobs

Looking for new proofs, new demonstrations involving already known mathematical results is the base of a mathematician's work and what makes its originality.

Pythagoras' theorem is useful for its practical aspects to build irrational lengths such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$... but it is also very much used nowadays by builders, architects and removal men.

KEYWORDS FOR WEBSITES:

Hypotenuse - Irrational numbers - Duplication - Squaring Pythagoras - Euclid...



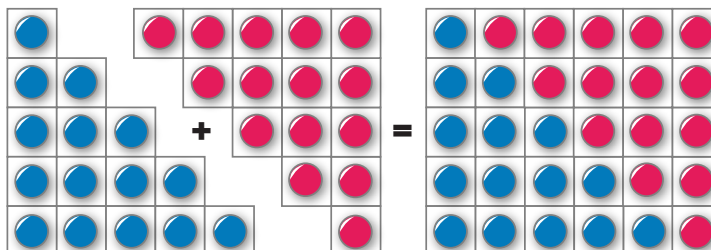


Do it yourself

MATERIAL: Round or square tokens

Triangular numbers: 1, 3, 6, 10, 15...

... are the sums of the first integers. Two ways of calculating these sums:



or:

$$\begin{aligned}
 &01 + 02 + 03 + 04 + 05 + 06 + 07 + 08 + 09 + 10 + 11 + 12 \\
 &+ 12 + 11 + 10 + 09 + 08 + 07 + 06 + 05 + 04 + 03 + 02 + 01 \\
 = &13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 + 13 \\
 = &12 \times 13
 \end{aligned}$$

therefore $T = \frac{12 \times 13}{2}$

And more generally prove that: $1 + 2 + 3 + \dots + n = \frac{n \cdot (n + 1)}{2}$

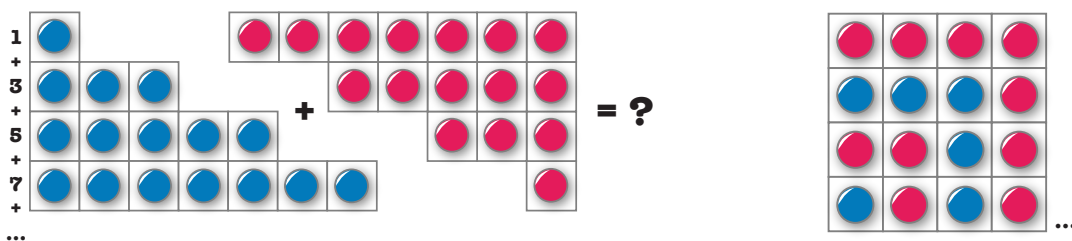
Check, on the first integers, that each of them can be written as the sum of a maximum of 3 triangular numbers.

To notice

From simple and easily understandable by schoolchildren, proofs have become much more complex and can sometimes be hundreds of pages long. For these proofs, it is necessary to use computers and only a limited number of specialists can read the results. **Pythagoras**, **Diophantus**, but **Pascal** too, studied these number sequences – represented by dots. In dimension 3, pyramidal, cubic... numbers can be found.

Square numbers: 1, 4, 9, 16...

... are the sum of the first odd numbers. Two ways to calculate these sums:

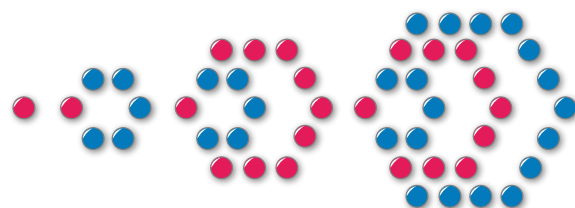
and more globally, prove that: $1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$

Check, on the first integers, that each of them can be written as the sum of a maximum of 4 square numbers.

Pentagonal, hexagonal... numbers

... are the sums of the first numbers represented on a pentagon, on a hexagon...

Find them and find their general formula. Check on the first integers that each of them is the sum of a maximum of 5 pentagonal numbers, of 6 hexagonal numbers...



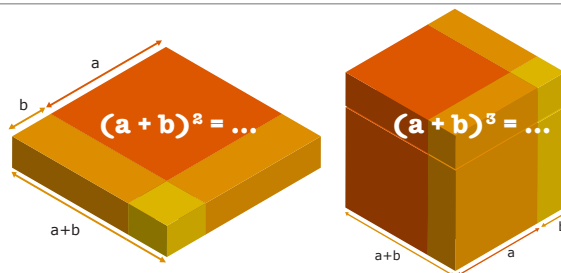


Do it yourself

MATERIAL: Wooden blocks or cardboard pieces

Algebra and Geometry

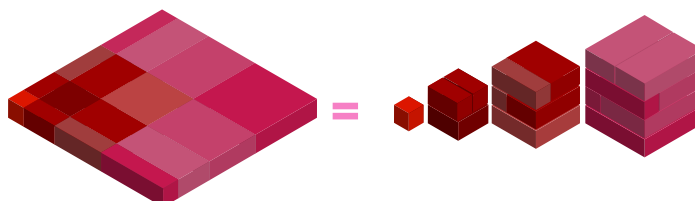
Use these blocks to find the formulas of:



and also, find the formulas of:

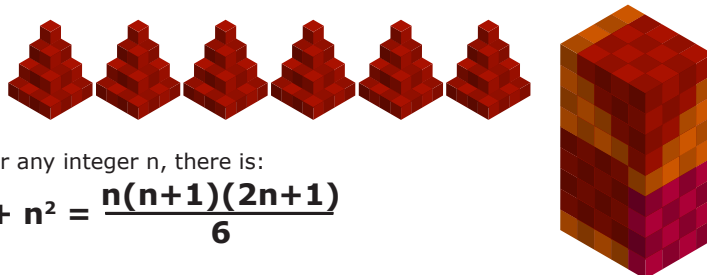
$$(a - b)^2 = \dots \quad a^2 - b^2 = \dots$$

Use these blocks to prove that:



$$(1 + 2 + 3 + 4 + \dots)^2 = 1^3 + 2^3 + 3^3 + 4^3 + \dots$$

With these 6 similar blocks, build a compact tower.

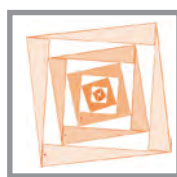
And more generally, prove that, for any integer n , there is:

$$1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Do it yourself

MATERIAL: Paper, 1 pen

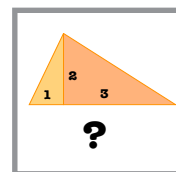
From conjectures to proofs



Pythagoras' triangles

Find the first right triangles where two legs are consecutive integers (such as 3 – 4 – 5). Try to deduce all the triplets which verify this property.

Fermat's squares

Find examples of the first prime numbers that can be written as the sum of two squares of integers (like $5 = 1^2 + 2^2$). From this reasoning, try to list all the primes that can be written in such a way.

Maths-related jobs

The main activity for scientists or mathematicians is to find and formulate a «good» problem.

But what is a good problem?

As **David Hilbert** said during the Second International Congress of Mathematics in 1900 in Paris – a good problem is a problem that even secondary schoolchildren can understand! But a good problem, even unsolved, should also be able to make knowledge progress. In fact, **Fermat's** last conjecture led the way to many solutions before it was demonstrated by **Andrew Wiles** 350 years later, in 1994.

KEYWORDS FOR WEBSITES:

Figurative numbers - Integer sequences - Conjecture



Do it yourself

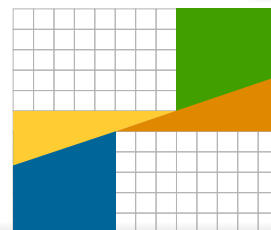
MATERIAL: Two 4-pieces jigsaws

115

$$63 = 64 = 65$$

Assemble the 4 jigsaw pieces into a square, then, into a rectangle.

- What is the area of the square? of the rectangle?
- What is the area of the surface at the beginning?

Question:Why are **Fibonacci** numbers involved?

Do it yourself

MATERIAL: a pattern to be reconstructed

116

Illusion or Reality?

Do these objects exist? Can you construct them? The answer is in a quotation of the famous french humorist Pierre Dac: «It's only a matter of point of view!»



Do it yourself

MATERIAL: Paper, 1 pen

From Pythagoras to Wiles

- Check that: $3^3 + 4^3 + 5^3 = 6^3$ $1^3 + 6^3 + 8^3 = 9^3$ $7^3 + 14^3 + 17^3 = 20^3$
- Find more examples of cube numbers which are the sum of 3 cube numbers.

To notice

Well before the time of **Pythagoras**, it was a known fact that some integers satisfy the following equation:

$$x^2 + y^2 = z^2$$

Find more examples other than (0,0,0) and (3,4,5).

In 1641, **Pierre de Fermat** wrote a **conjecture** stating that «it is impossible to separate a cube number, into two cube numbers», and that solutions with 0 and 1 are the only integers solutions of the following equation:

$$x^n + y^n = z^n$$

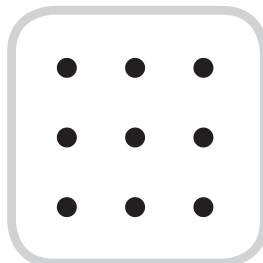
Before it was demonstrated by **Andrew Wiles** 350 years later, this conjecture was the topic of many others: is it correct? Is it unprovable? Undecidable?



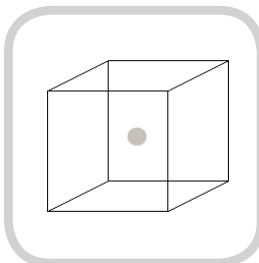
Do it yourself

MATERIAL: Paper, 1 pen, 6 matches

Change Point of View



- Join the 9 dots without lifting your pen and using only 4 straight lines.



- A hole was made in this transparent cube. What face(s) is it on?



- Use 3 matches to build a triangle, then, build 2 triangles with 5 matches, and finally, 4 triangles with 6 matches!



- Continue the sequence...

To notice

These questions used as psychological tests prove that in mathematics and by extension in life, it is essential to be "flexible" and to explore the various angles of a problem.

Maths-related jobs

Can we always prove something that we know is true?

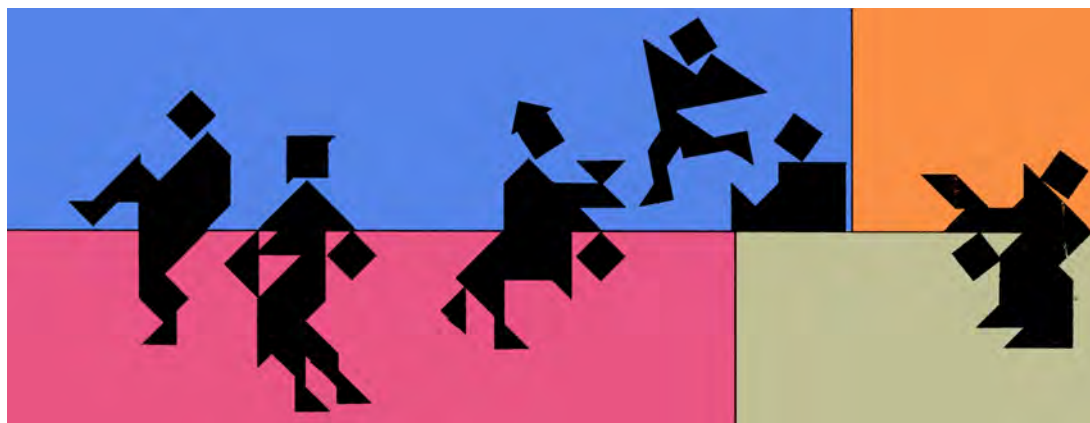
In 1931, **Kurt Gödel** answered in the negative with his famous «incompleteness» theorem.

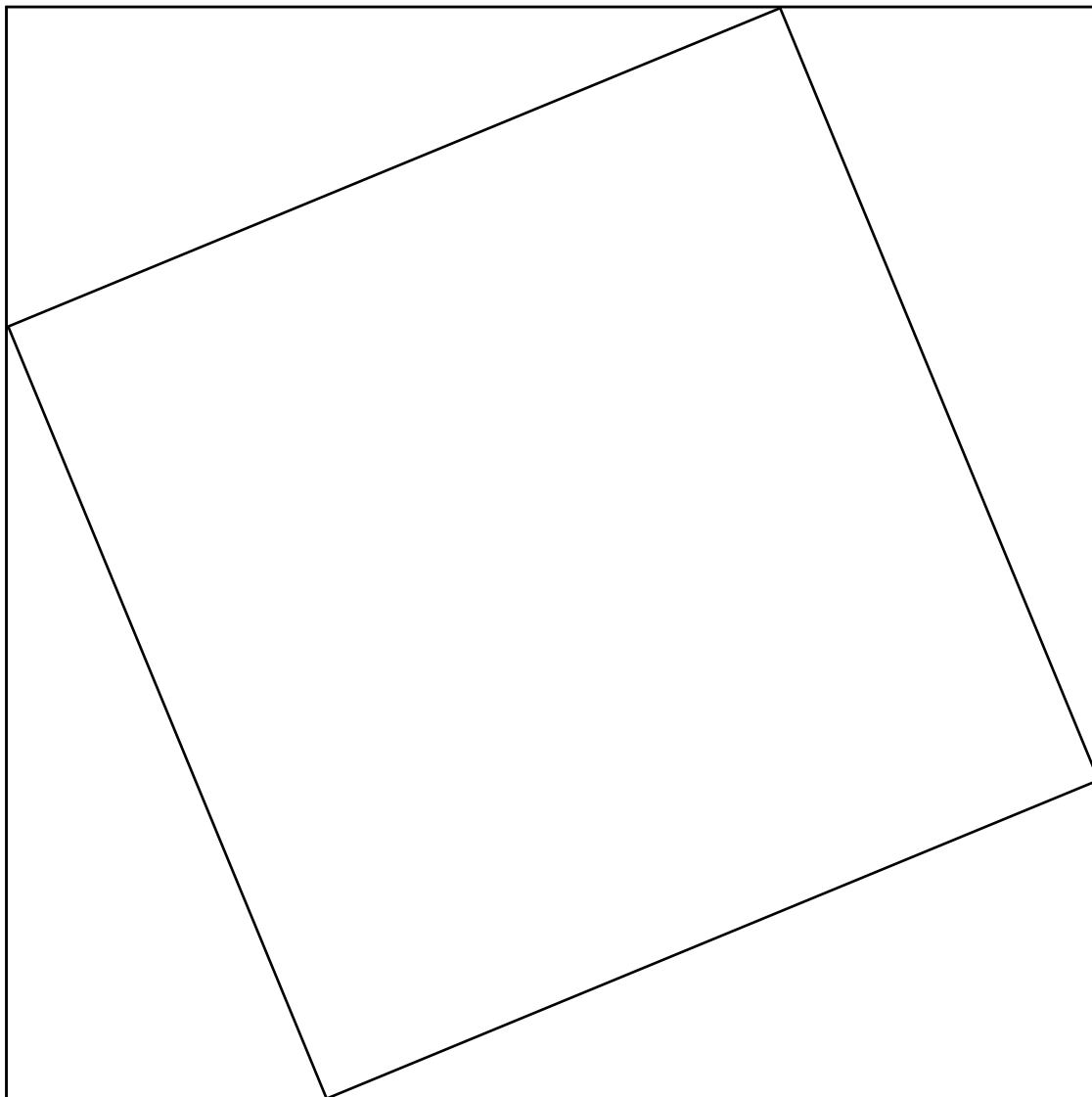
He proved that the notions of truth and provability cannot coincide by proving with an arithmetical example that a conjecture can be both true and impossible to demonstrate.

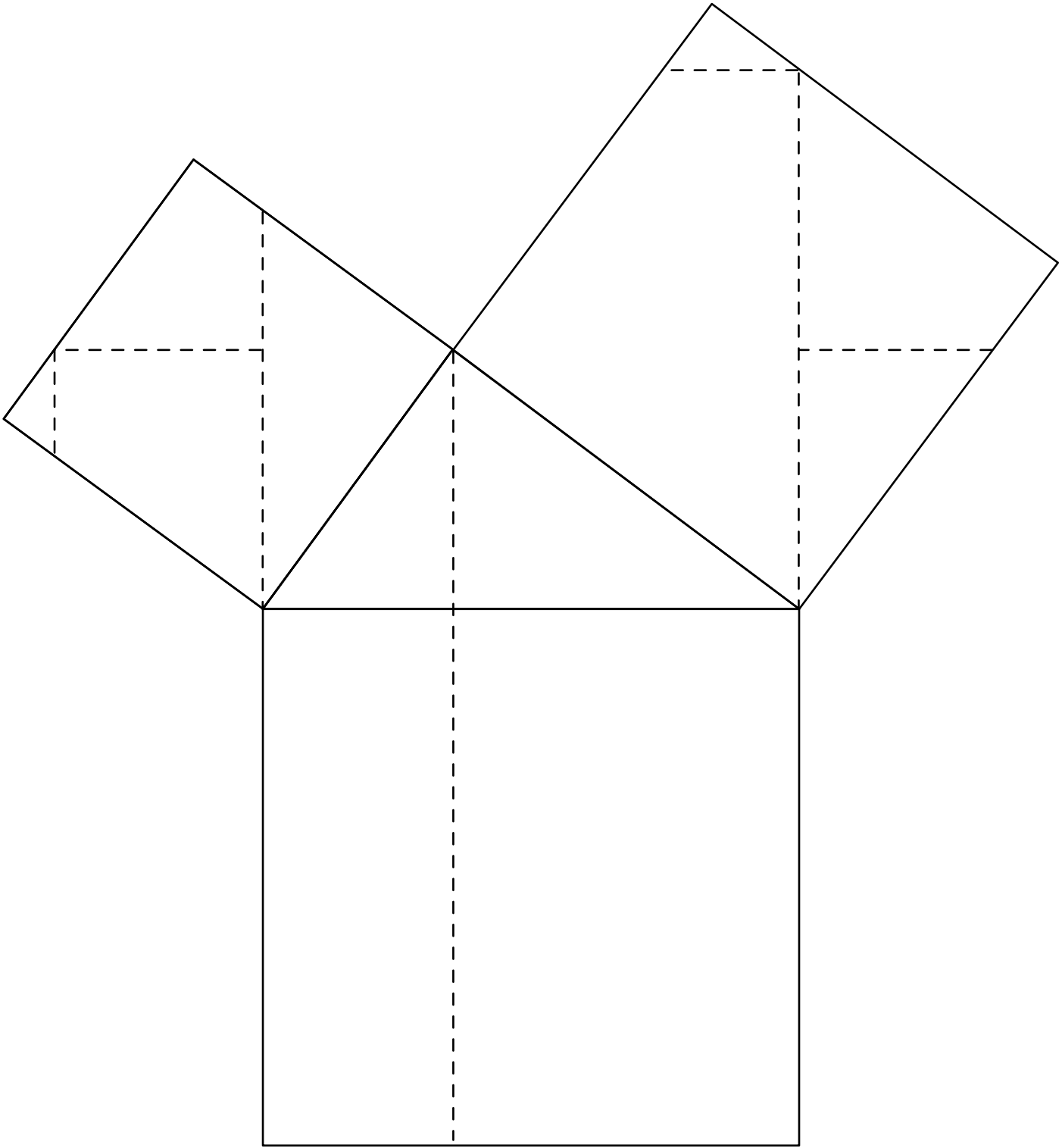
Problems said to have been undecidable up until now, in mathematics and computer science, are subjected to constant research.

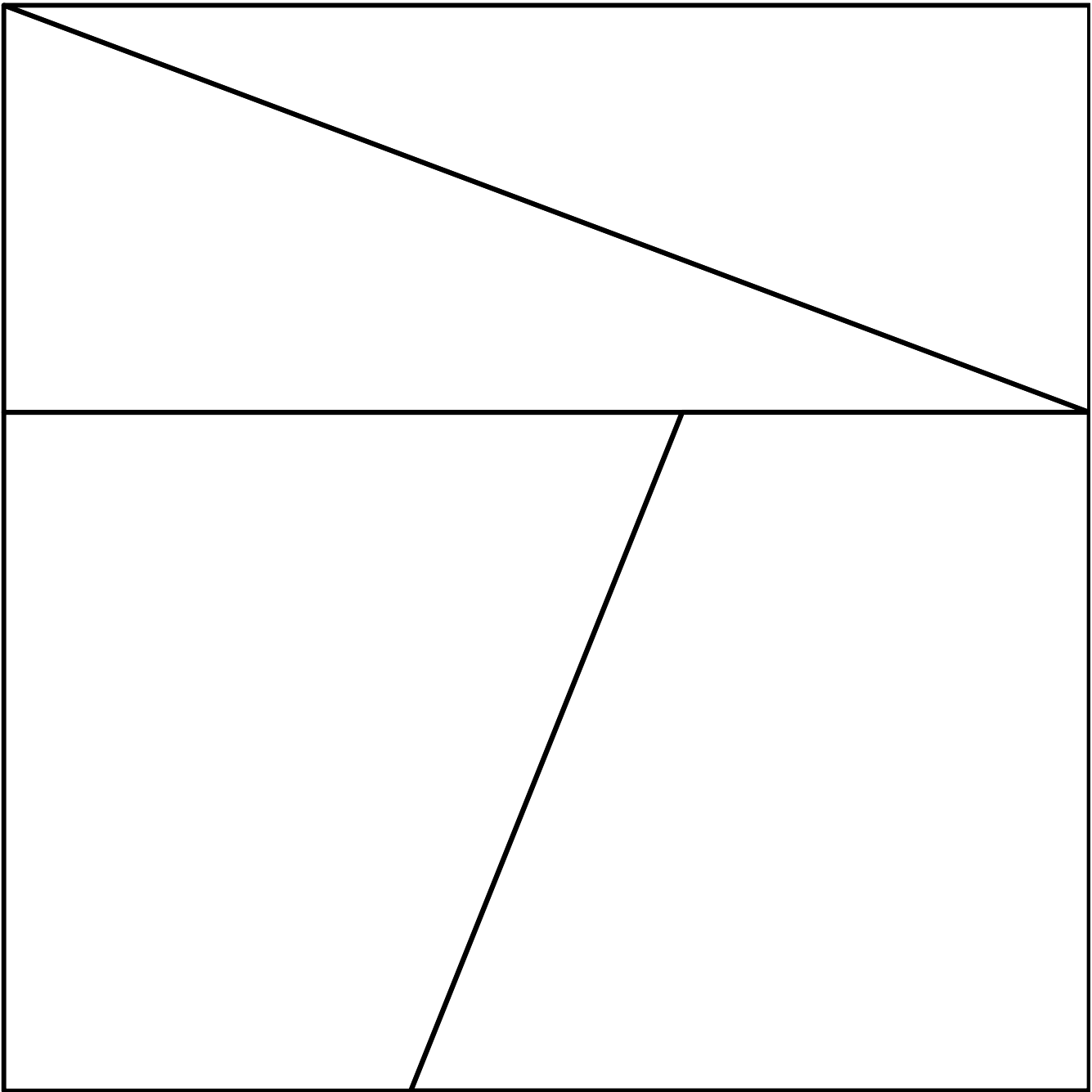
KEYWORDS FOR WEBSITES:

Undecidable - Calculable - Godel - P-NP Problems



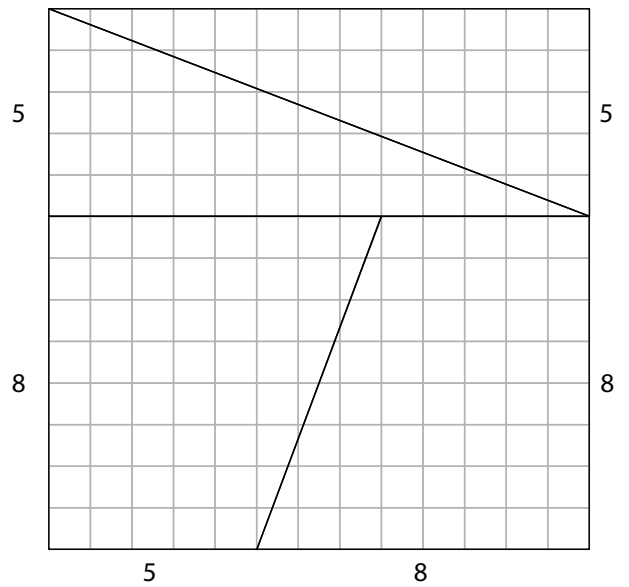
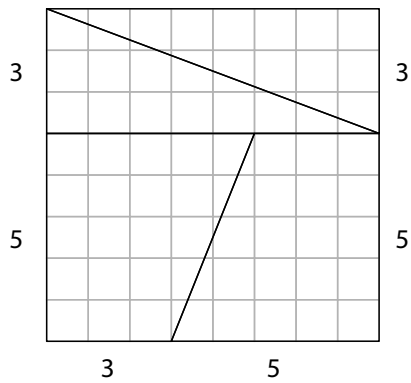


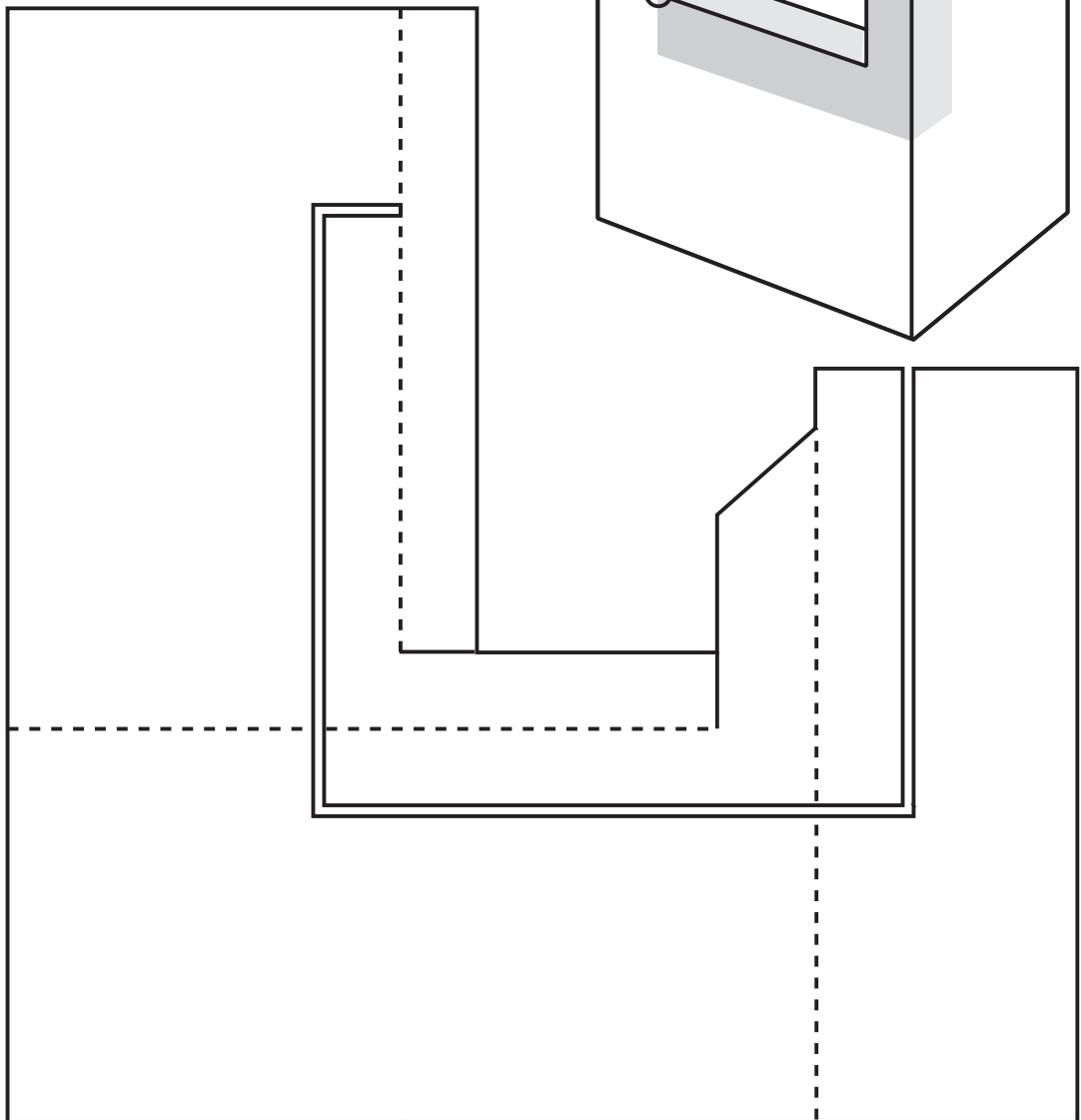
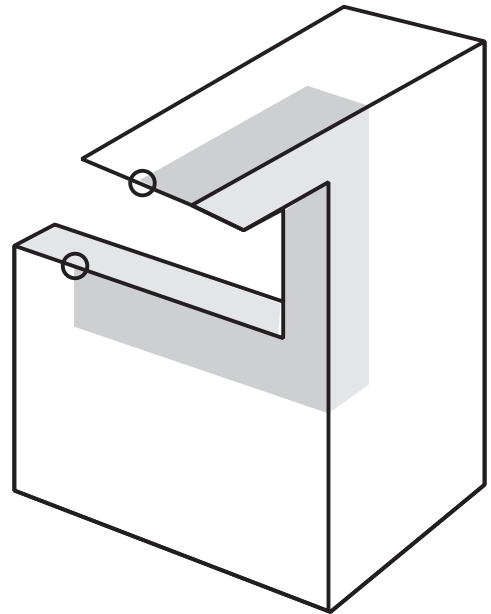




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10. Concluding



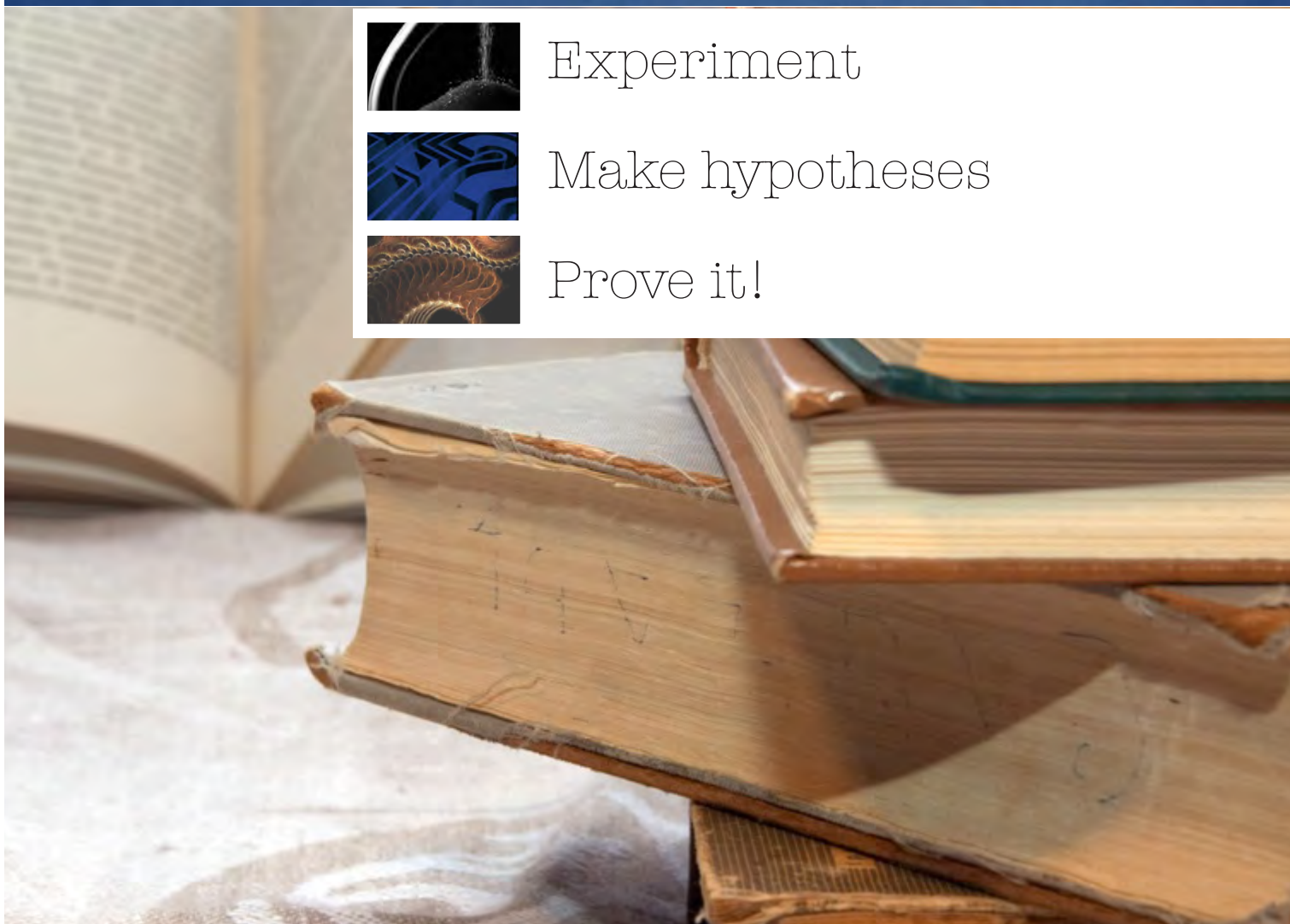
Experiment



Make hypotheses



Prove it!



10. Concluding Experiment



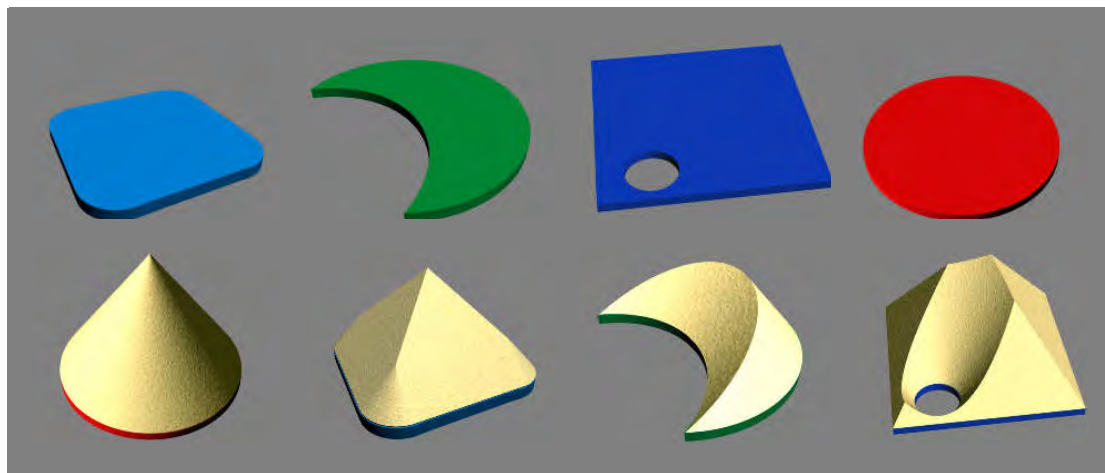
Do it yourself

MATERIAL: Cardboard polygons, 1 container with fine dry sand

Sand heap geometry

Choose a shape. Pour sand on until it spills over.

- When do you have a pyramid? A cone?
- Where is the summit?
- In the other cases, what shapes are the faces?
- How are the edges organised? What about the top edges?
- What happens if there is a hole in the shape?



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To Remember

All along this document, we gave you a set of situations for you and, above all, for your students to experiment. For these situations, we took inspiration from the works of mathematicians, physicists, mathematics teachers, educationalists and didacticians. We are very thankful to them. They were also taken from the exhibition called: "**Experiencing Mathematics!**", in French: « **Pourquoi les mathématiques ?** ». They can all be experimented with very little material. Many may also be extended with computers.

Do it yourself

MATERIAL: A4 drawing paper, **Folding models**

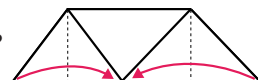
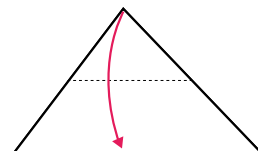
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122

123

Folds and Math

- What are the dimensions of a sheet of paper? Its mass?
- Fold the paper in two. What size is it now?
- Check that the new size is proportional to the previous one.
- Check that $L_4/L_4 = L_5/L_5 = \dots = \sqrt{2}$
- Does it work for any sheet of paper?
- Why does 80g paper mean? (or 120g, 150g, ... 250g)
- With these sheets, how can you make a square, an equilateral triangle...?
- Are they the biggest?



By folding paper, check that the sum of the angles of a triangle is equal to 180°.

- By the same, check that the bisectors of a triangle meet in a single point.
- Do the same for the heights and right bisectors...
- How do you make a regular pentagon, hexagon?



Maths-related jobs

One of the approaches of the scientist is the **experimental approach**. It may also be that of the mathematician who tries to find answers after having raised a – good- question. However, in some domains such as astronomy, particle physics, biology, the **experimental approach** is almost impossible to use for practical, ethical reasons...

The approach of the mathematics teacher is to learn how to give his students experimental situations where each of them can take a question in, make it his and look for answers.

The approach of the math student is to learn how to formulate hypotheses, probe them and finally demonstrate that they are true... or false.

KEYWORDS FOR WEBSITES:

Experimental approach - Scientific methods



To Remember

Little questions, great problems?

Mathematics have always offered funny or astonishing wordings. Among all these recreative problems, some ask pertinent questions. This means that looking for the answers, or the answers themselves, allows to highlight a mathematical or scientific facts or even an important reasoning technique.

A good question may also be a question which enables the students detect the **implicit models** used by the students in order to show them the inadequacy.

Reasoning is what these questions are about: conservation of quantities for the water and wine experiment, topology for the strip, modelisation for the coffee, sampling for the rice, logics for the burning sticks...

Going further

Little questions, great problems



Water in the wine

Two glasses hold the same volume of liquid, water in one, wine (or coffee...) in the second. Take a spoonful of water, pour it into the wine and stir. Now, take a spoonful of the mixture and pour it into the water.

- Is there more water in the wine or wine in the water?
- How can you prove your hypothesis?



Möbius strip

With a paper bracelet, make three bracelets with two cuts of scissors. Take a strip of paper and make a «Möbius» strip by half-twisting one end.

- How many cuts does it take to have 3 strips?



Prime knots

Tie each of these knots.

- What is the minimum number of crossings?
- Try to find the first knots with 3, 4 ... crossings.



One kilo of coffee?

1 kilo of 1 cm marbles; 1 kilo of 1 mm marbles. Place each type of marbles in a cubic container.

- Which kilo occupies the lesser volume?
- What is the link with the question about 1 kilo of coffee?



One kilo of rice!

- How many grains of rice are in one kilo?
- Find at least 2 ways to calculate the answer.



One hour!

2 different sticks (or candles) burn in one hour.

- How can you measure 1/4h? 1/2h? 3/4h?
- Find at least one method.

Maths-related jobs

*Is knowing being aware of one's ignorance?
(in Meno - Plato)*

In secondary school - but also later - students show poor **debating** and convincing skills in mathematics, using logical and mathematical arguments.

For that, they must learn how to express the various points of their argumentation and to explain the logical relations which led to their conclusions.

Knowing how to prove is learning how to debate using well-formulated mathematical arguments. Knowing how to prove is also knowing what a proof is, make it one's own and use it appropriately.

KEYWORDS FOR WEBSITES:

Debating - Method - Logics - Aristotle - Plato - Descartes

10. Concluding

Prove it!



Do it yourself

MATERIAL: Cardboard to cut out, flip book to make

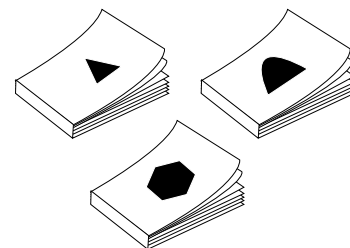
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Flatland: a 2D world

Who am I?

Each book shows the marks left by objects as if seen through a flat screen.

- Associate an object to each mark.

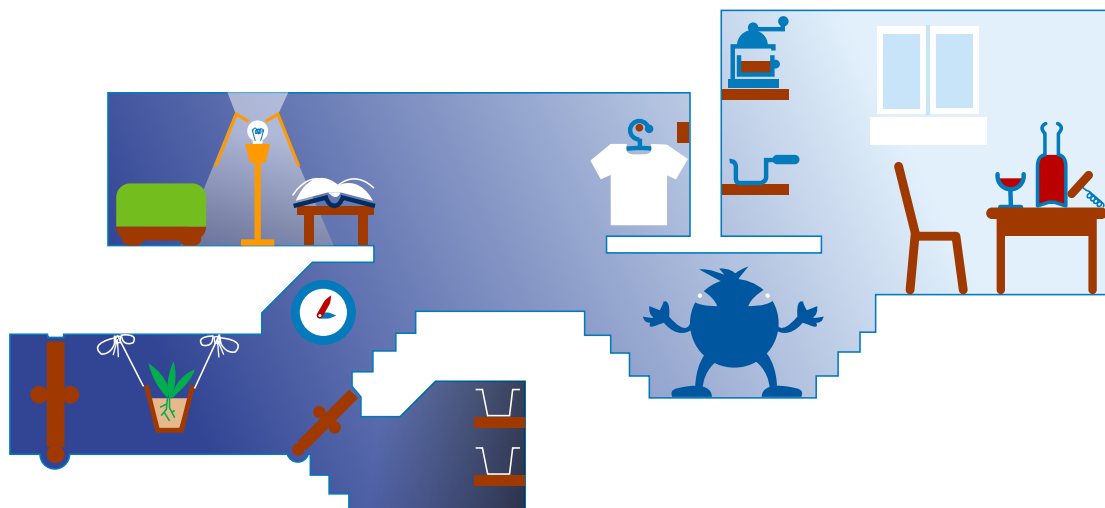


Going further

Life in Flatland

The drawing shows the house of an alien living on an imaginary planet where there would only be 2 dimensions and everything would be flat.

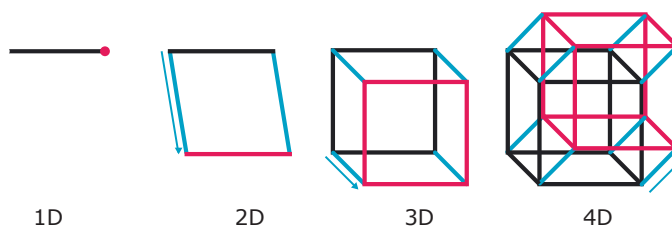
- Find 7 errors.
- Imagine other objects which could exist, or not, in this flat world.



To remember

Giving the students an experimental approach in mathematics is more than just manipulating geometrical objects, numbers, letters. These experiments must give them the opportunity to take the situation in (**experimentation phase**), to express hypotheses out loud or on paper (**formulation phase**) and to finally validate the hypotheses (**validation phase**). In the experimental approach, teachers will have **to summarize and highlight the mathematical knowledge**, the new facts, the new reasoning methods **acquired by the students**.

Towards the 4th dimension



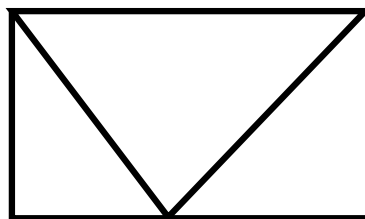
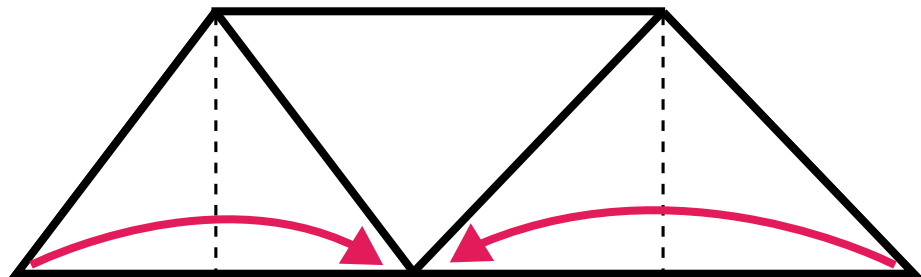
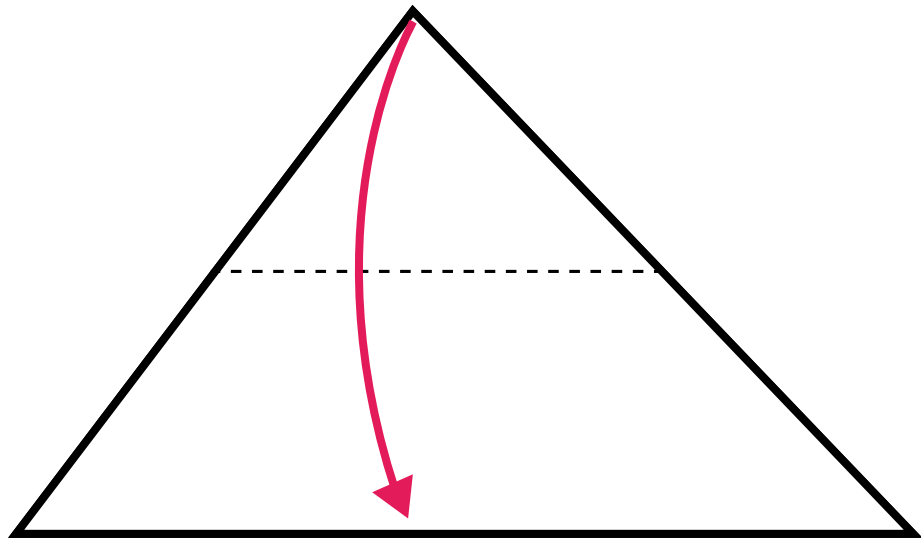
Maths-related jobs

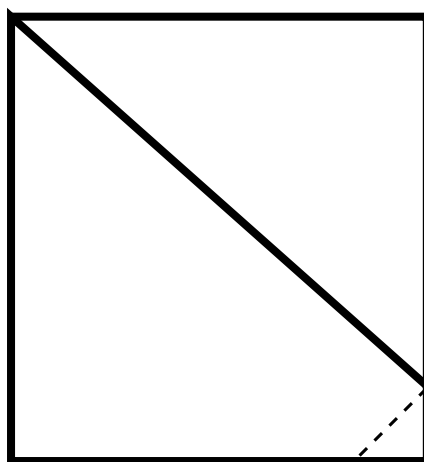
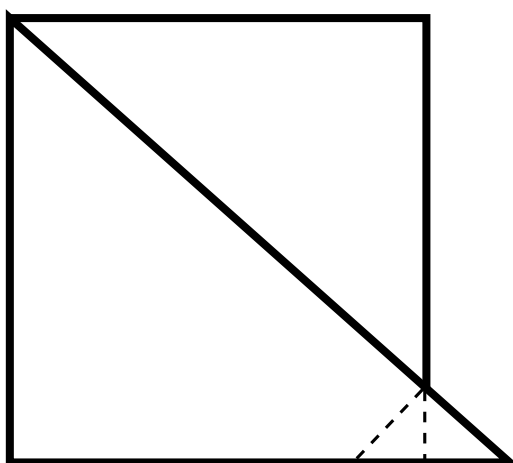
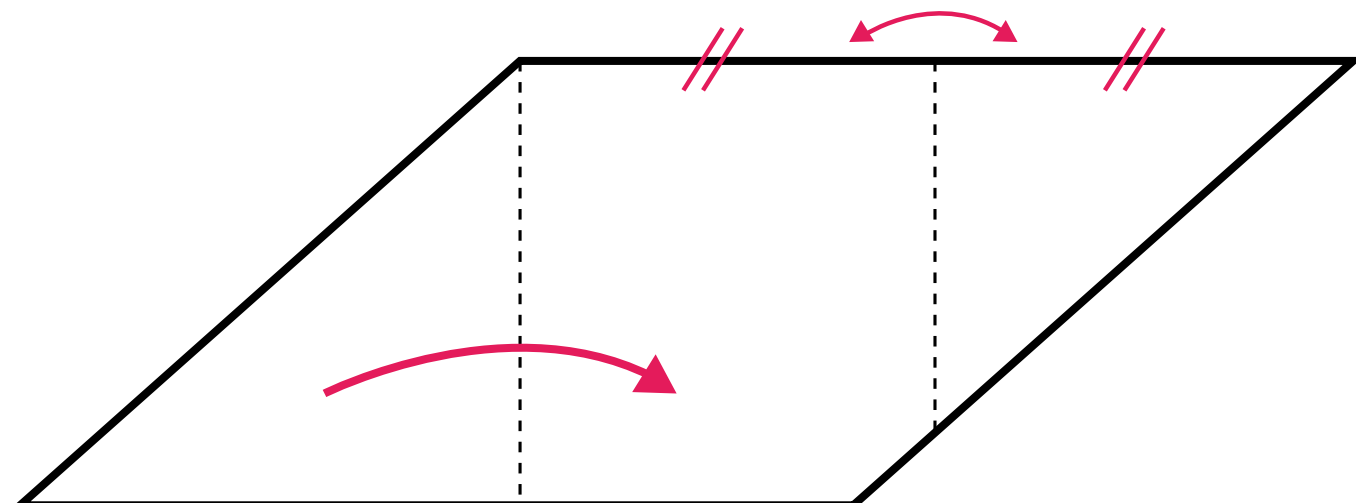
Math teachers, just like mathematicians, physicists or any other scientist, must learn how to prove, and above all must teach their students methods how to prove, how to debate and bring forward arguments based on mathematical tools: mathematical facts and mathematical logics.

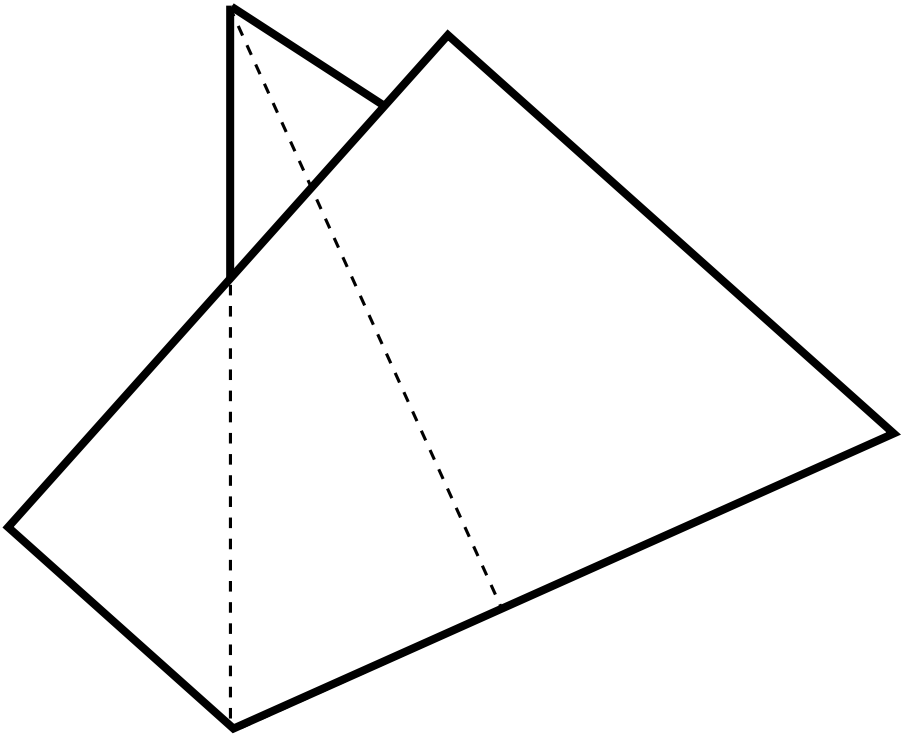
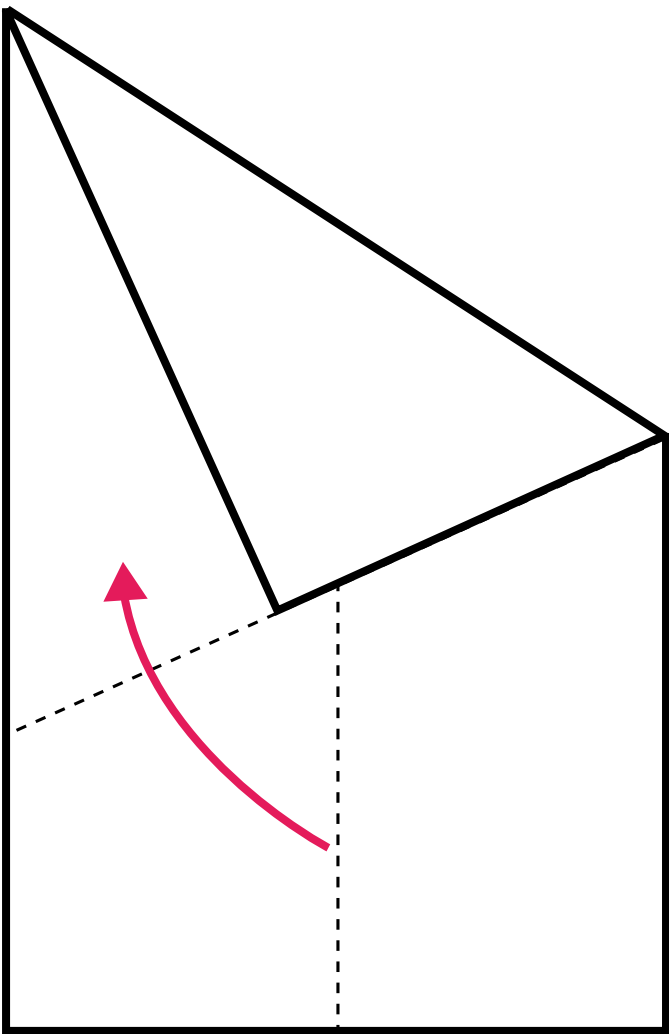
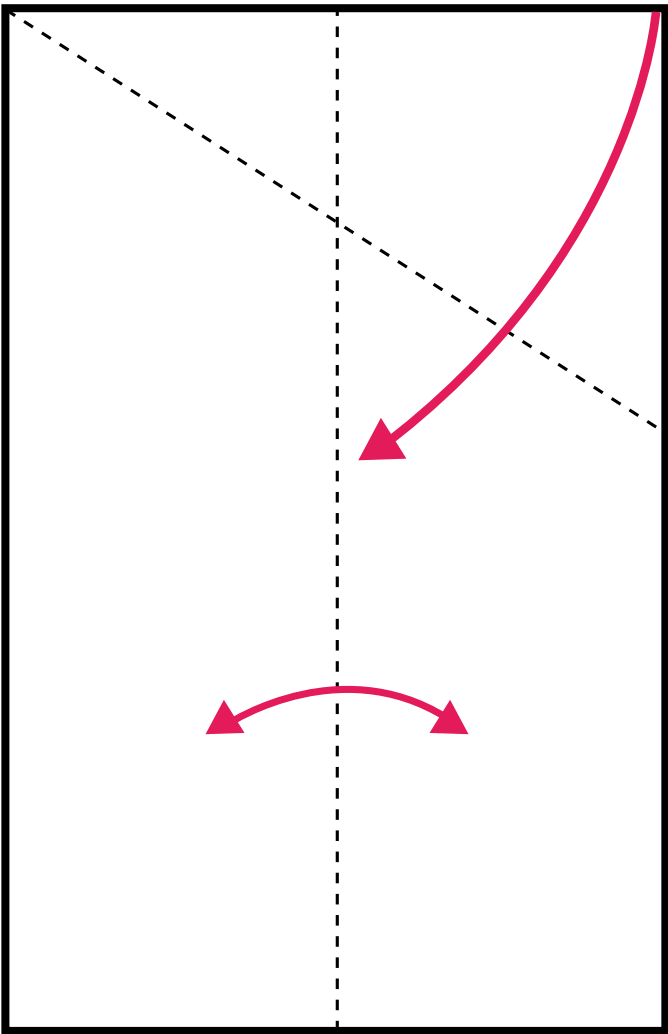
Learning to prove that a solution exists or not, that a statement is true or false, or even that it is not possible to prove it, at least with the current knowledge of the student, is also part of the teachers' duties.

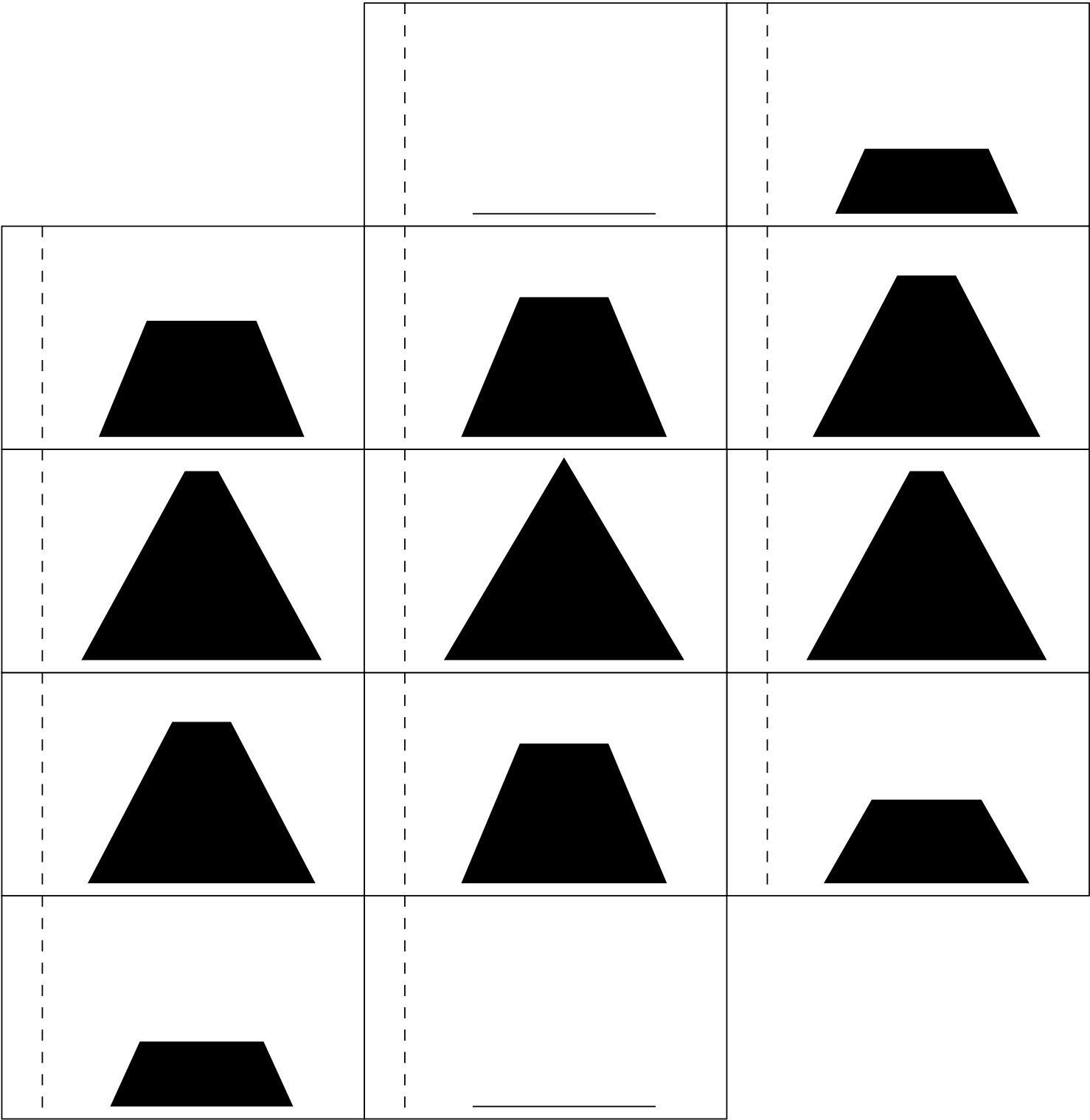
KEYWORDS FOR WEBSITES:

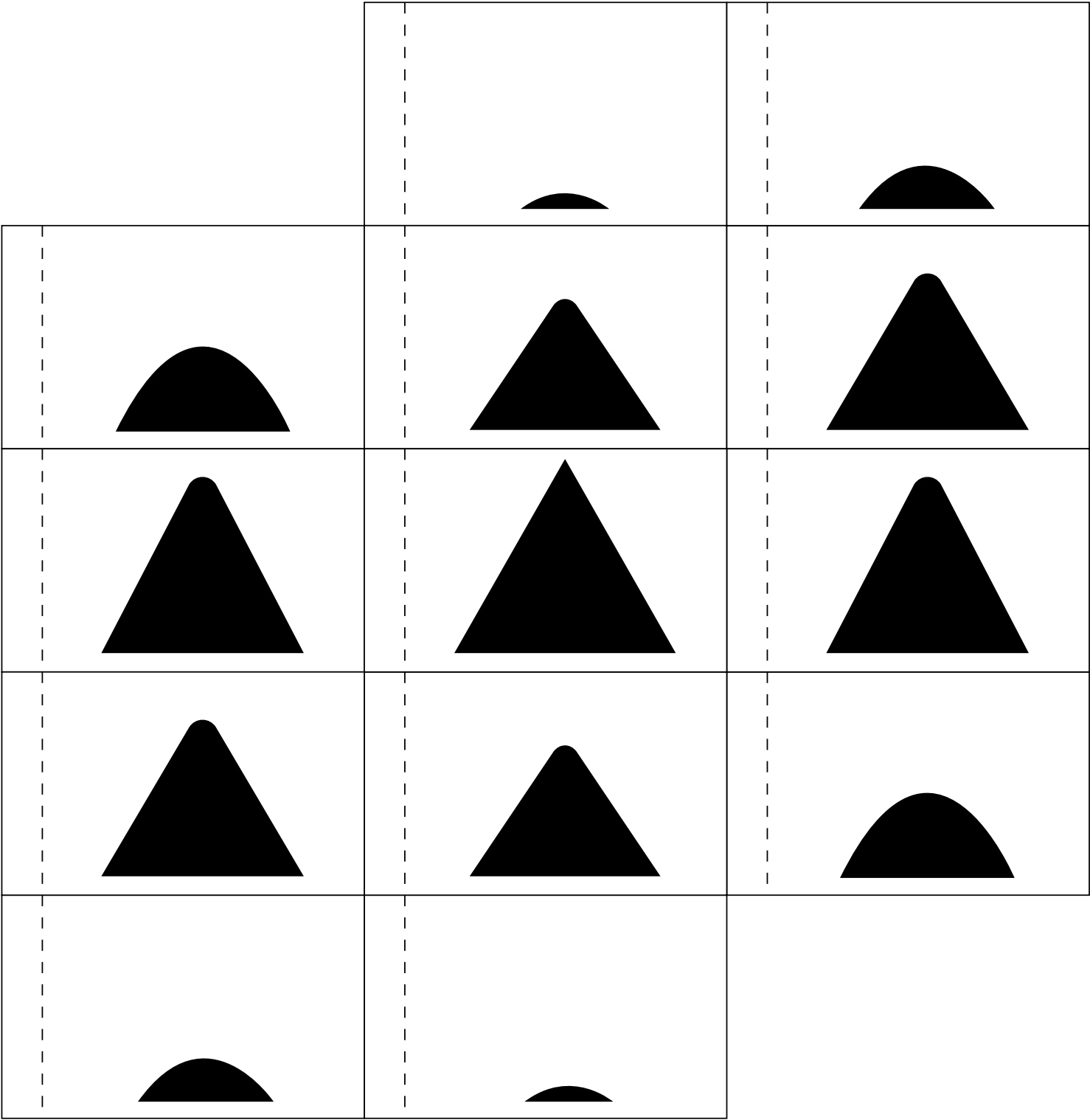
Mathematics - Mathematics teacher - Mathematician

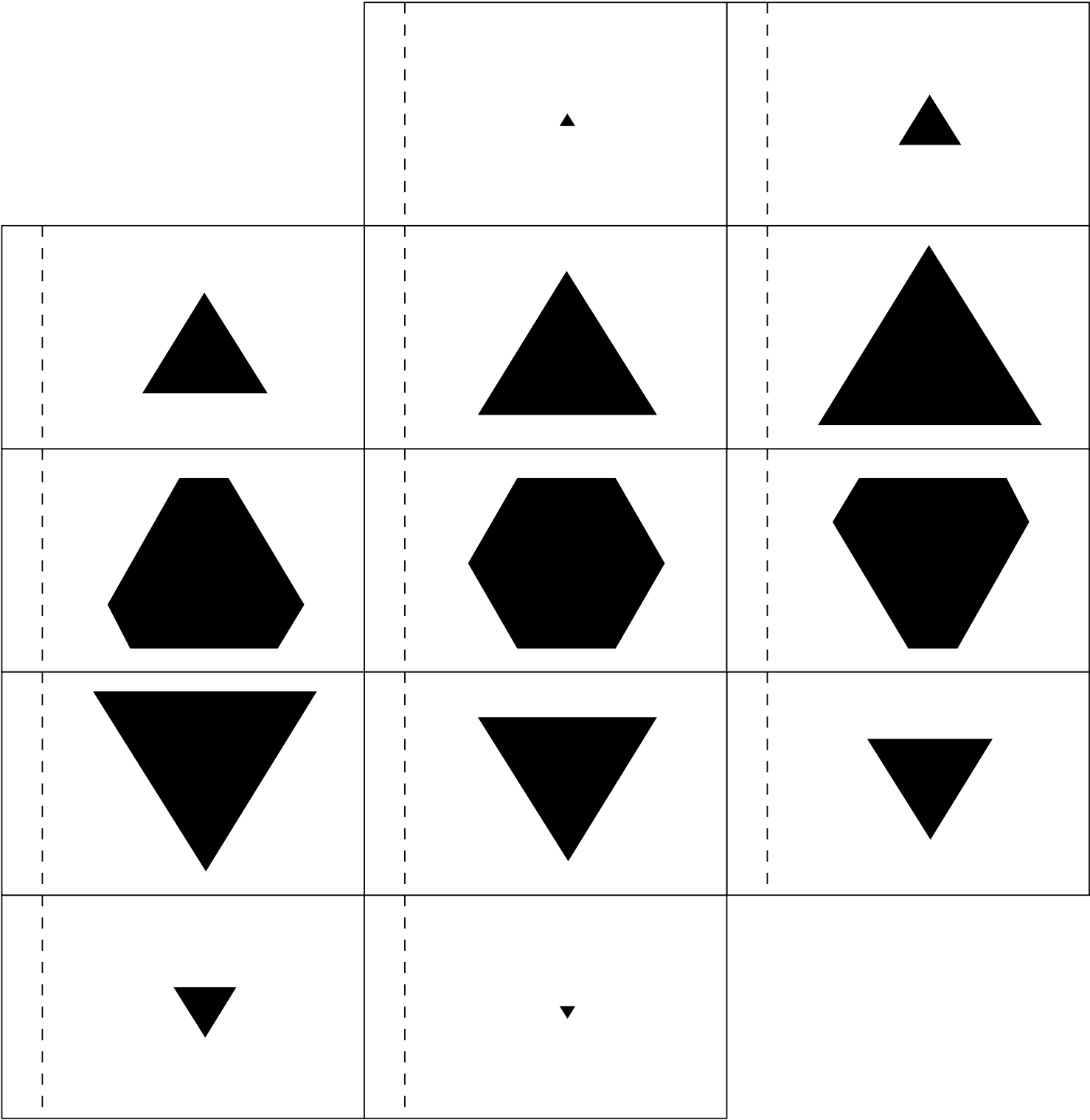














credits

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with the collaboration of:

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 Mari-Jo Ruiz, Ateneo de Manila University, Philippines
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- Pt: Bento Louro – Universidade Nova de Lisboa
- Ct: Armengol Gasull & Gregori Guasp – Universitat Autònoma de Barcelona
- Ar: Boubaker-Khaled Sadallah - Ecole Normale Supérieure - Algiers-Kouba

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